Winding Numbers and Toeplitz Operators

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Winding Number

Definitions

Path: A continuous function from the unit interval to complex numbers.

Loop: A path $f(t)$ such that $f(0) = f(1)$.

Exponential Lift of $f(t)$: A function $g(t)$ such that $e^{g(t)} = f(t)$.

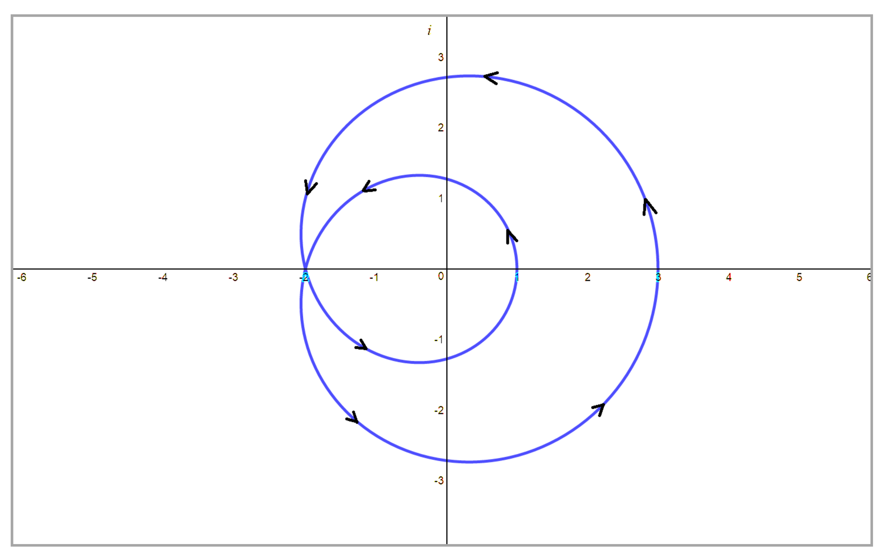
If a loop $f(t)$ has exponential lift $g(t)$, then $g(0) = g(1) + 2\pi in$ for some integer n. This number is the winding number of $f(t)$. Intuitively, the winding number of a loop is the number of times the function loops around the origin, counterclockwise. Winding numbers relate to many areas of Math. They can be used to count roots of polynomials and are used in Cauchy's residue theorem to compute line integrals.

Winding number can be calculated with the formula $\frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(t)}{f(t)}dt$.

Properties:

Winding number of $(f(t)g(t))$ = Winding number of $f(t)$ + Winding number of $g(t)$.

If $f(t)$ is homotopic to $g(t)$ through loops, then Winding number of $f(t)$ = Winding number of $g(t)$.

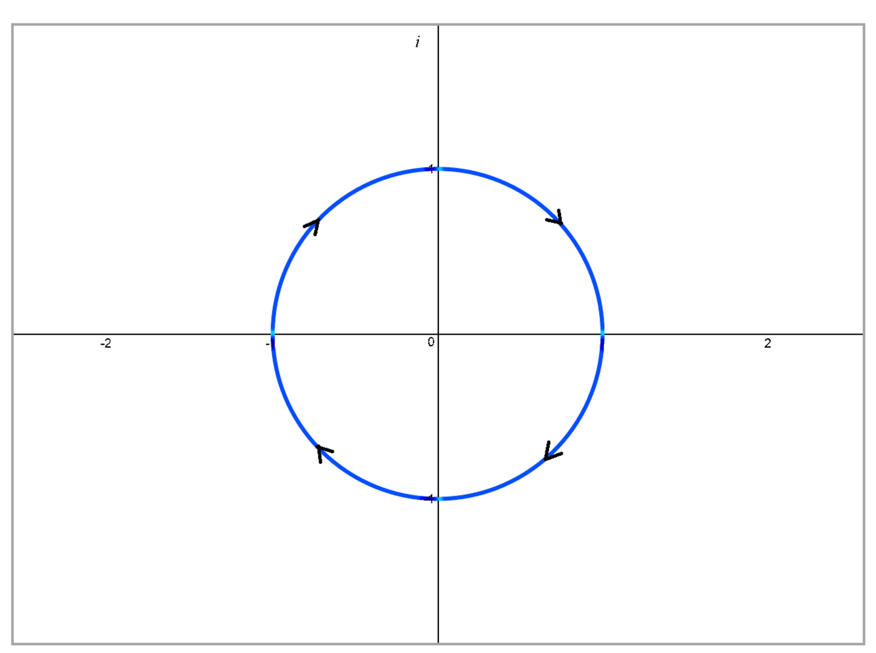


A graph of a path with winding number 2.

Loops and Continuous Functions on the Circle

Loops can be considered as the image of a continuous functions on the unit circle, with z = 1 as the starting and ending point.

$C(T)$ (continuous functions on the circle) has an orthonormal basis of all integer powers of z. The winding number of z^n is n.



A graph of z^{-1} with winding number -1.

Hardy Space and Toeplitz Operators

Definitions:

The Hardy Space H^2 is the functions in $C(T)$ formed from the basis of z^n with $n \geq 0$.

A Toeplitz Operator T_f maps a function in H^2 to another function in H^2 by the following formula:

$T_f(g) = P(fg)$ where P is the projection onto H^2.

Example: $T_{z^{-1}}(3 + z - z^2) = P(z^{-1}(3 + z - z^2)) = P(3z^{-1} + 1 - z) = 1 - z$

Compact Operators

Definitions:

Compact Operator: An operator that maps the unit ball to a set with compact closure.

Symbol Map: The map sending the Toeplitz Operator T_f to $f(z)$.

The Toeplitz Operators with continuous symbol form an Algebra $\mathcal{T}$. In this Algebra, any semi-commutator $T_{fg} - T_f T_g$ is a compact operator. The compact operators $\mathcal{K}$ form a sub-algebra and give rise to the following short exact sequence:

$$0 \to \mathcal{K} \to \mathcal{T} \to \mathcal{T} / \mathcal{K} \to 0$$

$\mathcal{T} / \mathcal{K}$ is the quotient algebra in which two elements $T_f + \mathcal{K}$ and $T_g + \mathcal{K}$ are equal if there is some compact operator V such that T_f + V = T_g. Using the fact that the symbol map is onto $C(T)$ and the functional calculus, it can be shown that $\mathcal{T} / \mathcal{K}$ is isomorphic to $C(T)$ and that

$$0 \to \mathcal{K} \to \mathcal{T} \to C(T) \to 0$$

is also a short exact sequence.

Index and Winding Number

Definitions:

Kernel of T_f: The set of functions in $C(T)$ such that $T_f(g) = 0$.

Index of T_f: The dimension of the kernel of T_f minus the dimension of the kernel of the adjoint of T_f which we will label $(T_f)^*$.

Example. The index of T_z = -1 as the kernel of T_z is empty, while the kernel of $(T_z)^*$ = $T_{z^{-1}}$ contains 1. (As $P(z^{-1} * 1) = P(z^{-1}) = 0$)

Similarly the index of $T_{z^n} = -n$.

Further, the index of $T_f T_g$ = index T_f + index T_g, if two Fredholm operators are homotopic, they have the same index, and if T_f + V = T_g for a compact V, then the indices of T_f and T_g are the same.

An argument from the above properties concludes that the winding number of $f(z)$ = the negative of the index of T_f: our formula for winding number can be used to calculate index!

Extending the Toeplitz Algebra

There are many possible extensions of the algebra of Toeplitz operators with continuous symbol. One such extension will contain Toeplitz operators with symbols in $C(T)\times_\alpha \mathbb{Z}_2$. The action for this crossed product algebra is denoted α with $\alpha(f(z)) = f(-z)$. Thus $T_{f+g\alpha}(h(z)) = P(f(z)h(z)+g(z)h(-z))$.

It is natural to try to establish a connection between the indices of these operators and some formula similar to the one for winding number.

Results

By an argument similar to the one used in the previous case, semi-commutators can be shown to be compact. Again, this gives rise to a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T} \to \mathcal{T}(C(T)\times_\alpha \mathbb{Z}_2)/\mathcal{K} \to 0$$

Since the index of $T_{z^n} = -n$, and semi-commutators are compact, the index of $T_{az^n+z^m\alpha}$ can be calculated by finding the index of $T_{a+z^{m-n}\alpha}$.

By direct calculation of the elements in the kernel of $T_{a+z^{m-n}\alpha}$ and its adjoint, it is found that the index is 0 if $|a| > 1$ and $n - m$ if $|a| < 1$.

By previous index properties it follows that index of $T_{az^n+z^m\alpha} =$ index of $T_{z^n}T_{a+z^{m-n}\alpha}$ $=$ index of T_{z^n} + index of $T_{a+z^{m-n}\alpha} = -n$ if $|a| > 1$ and $-m$ if $|a| < 1$.

Up Next:

The index for a general element in the crossed product algebra will need to be established. A formula, similar to the one for winding number, for elements in the crossed product algebra will be used. Then the same connection between this formula and index will be established as it was for the regular Toeplitz algebra.

Other crossed product Toeplitz algebras, based on other finite groups, will then be investigated.