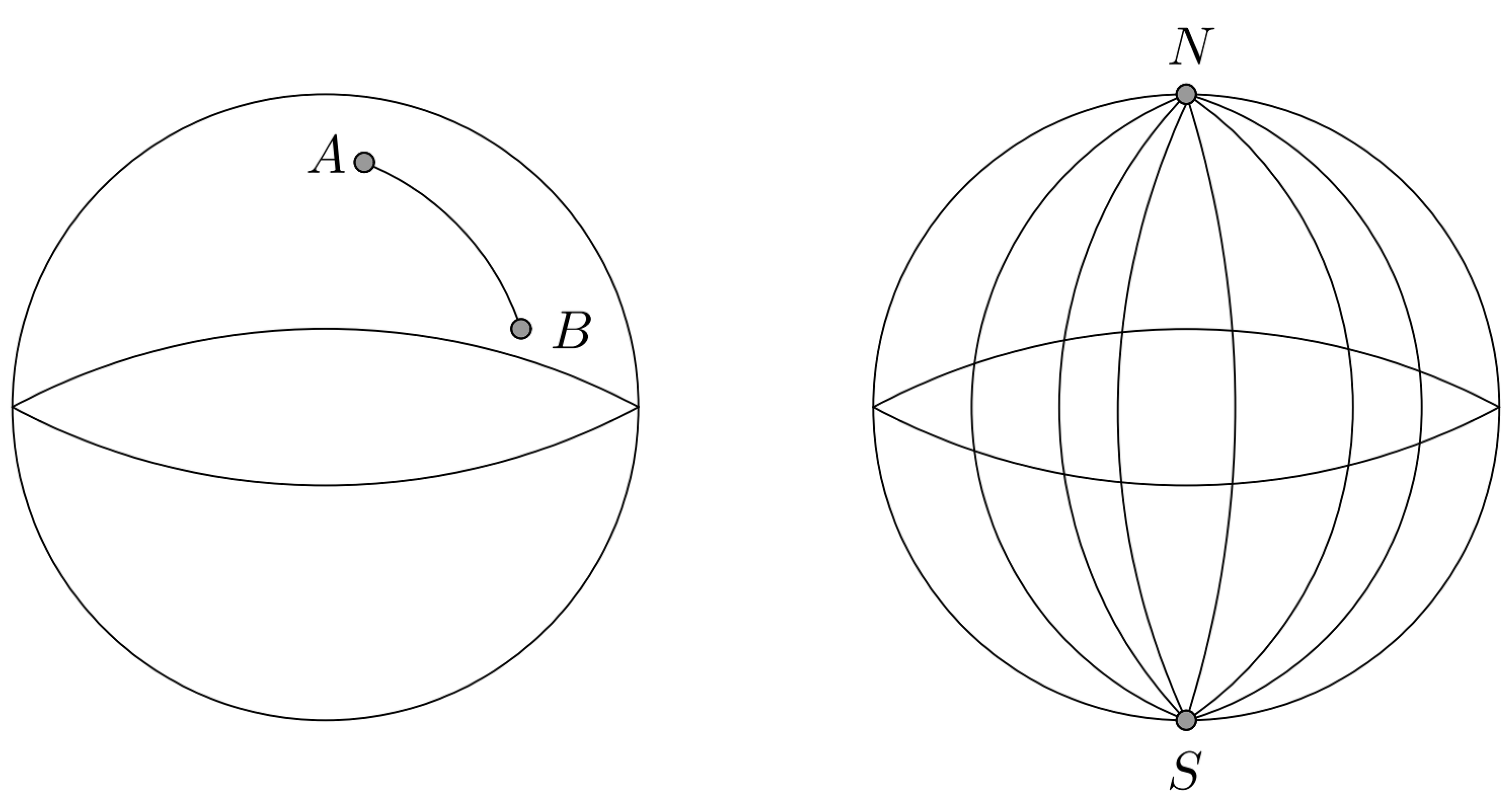
Small Eigenvalues of Hyperbolic Polygons

Thinh Doan, Advisor: Ken Richardson
Department of Mathematics - Texas Christian University

Abstract

The hyperbolic plane is a type of non-Euclidean geometry of the plane where a different distance formula from the standard one is used.Hyperbolic geometry has a profound impact on various distinct fields of mathematics and modern physics such as the study of complex variables, geometric group theory, topology and theory of special relativity. Using the finite element method as a numerical approximation in solving for eigenvalues of the hyperbolic Laplacian, this research investigates the estimates of the first two eigenvalues with Dirichlet and Neumann boundary conditions on bounded domains in the upper half plane. These values can further be used to shed light on the Selberg and Fundamental Gap conjectures.



Spectral Theorem

Spectral Theorem:If A is a symmetric operator on the finite dimensional vector space V with complex inner product, then

1. All eigenvalues of A are real
2. A is diagonalizable, and
3. You can choose eigenvectors of A so that they form an orthonormal basis of V,
4. Eigenvectors corresponding to different eigenvalues are automatically orthogonal.

A similar result is valid for symmetric elliptic differential operators like the Laplacian on infinite dimensional Hilbert Space

The Laplacian

The Laplacian measures the curvature or stress of a function. It tells you how much the value of the function differs from its average value taken over the surrounding points.More precisely, it is the divergence of the gradient.
Laplacian $= \Delta f = -\nabla \cdot \nabla f$
The operator depends on the geometry of the underlying manifold. The Laplacian on $\mathbb{H}^2$ (upper half plane model) is

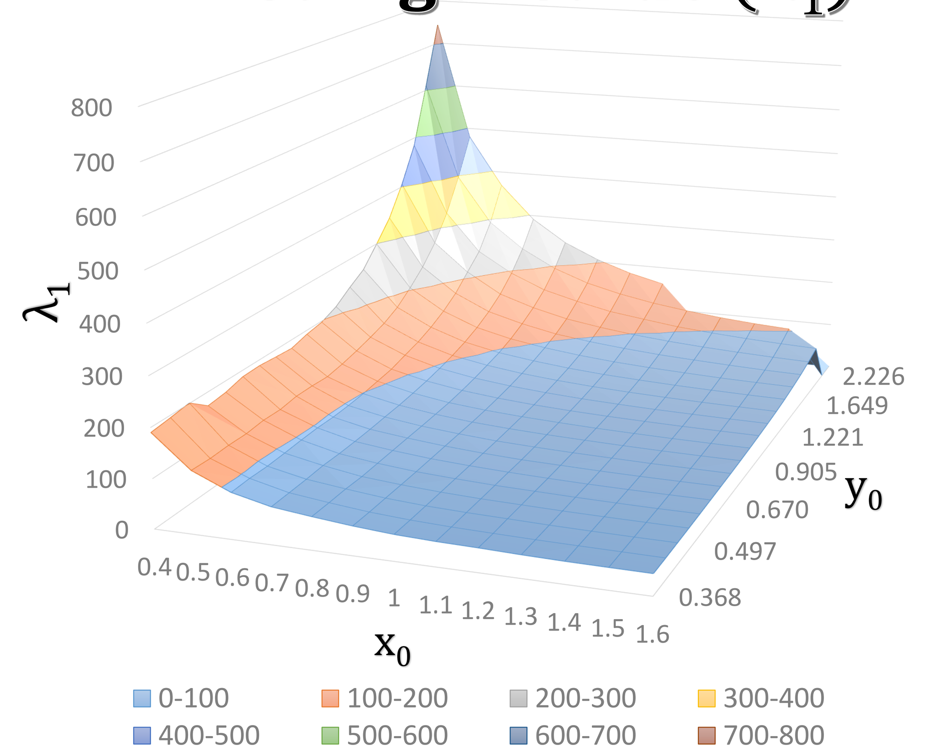
$$(-y^2)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = -4(Im(z))^2 \frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}.$$

Methods for estimating the first two eigenvalues

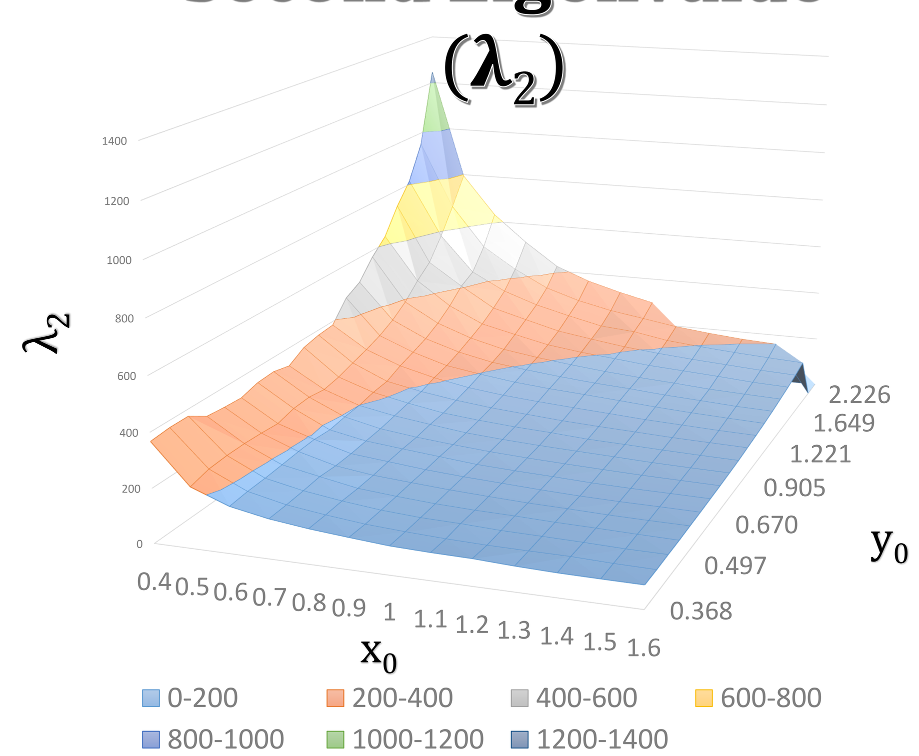
Given a polyhedron in $\mathbb{H}^2$, we have various boundary conditions for functions ψ defined on the polyhedron:

1. Dirichlet boundary condition: $\psi(z) = 0$ when z is on boundary.
2. Neumann boundary condition: $\frac{\partial}{\partial n}\psi(z) = 0$ when $z \in$ boundary. Here $\frac{\partial}{\partial n}$ means the outward normal derivative at points of the boundary.
3. Periodic boundary conditions on a polygon is equivalent to considering functions on $\mathbb{H}^2$ that satisfy a periodicity condition.

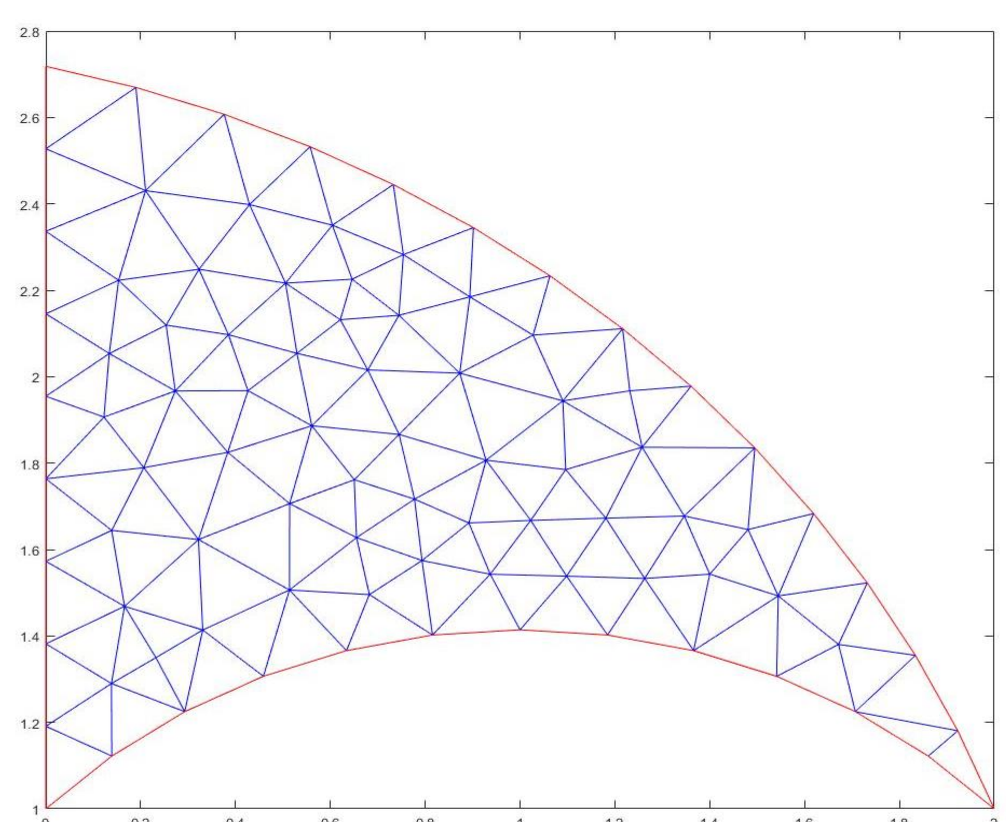
Given a hyperbolic triangle $\triangle ABC$ with $A = (x_1, y_1) = (0, e)$, $B = (x_2, y_2) = (0, 1)$, and $C = (x_0, y_0)$. By changing the coordinates of C (i.e. x_0 and y_0), and using Dirichlet boundary condition, we can estimate the change in the first two eigenvalues and their difference.



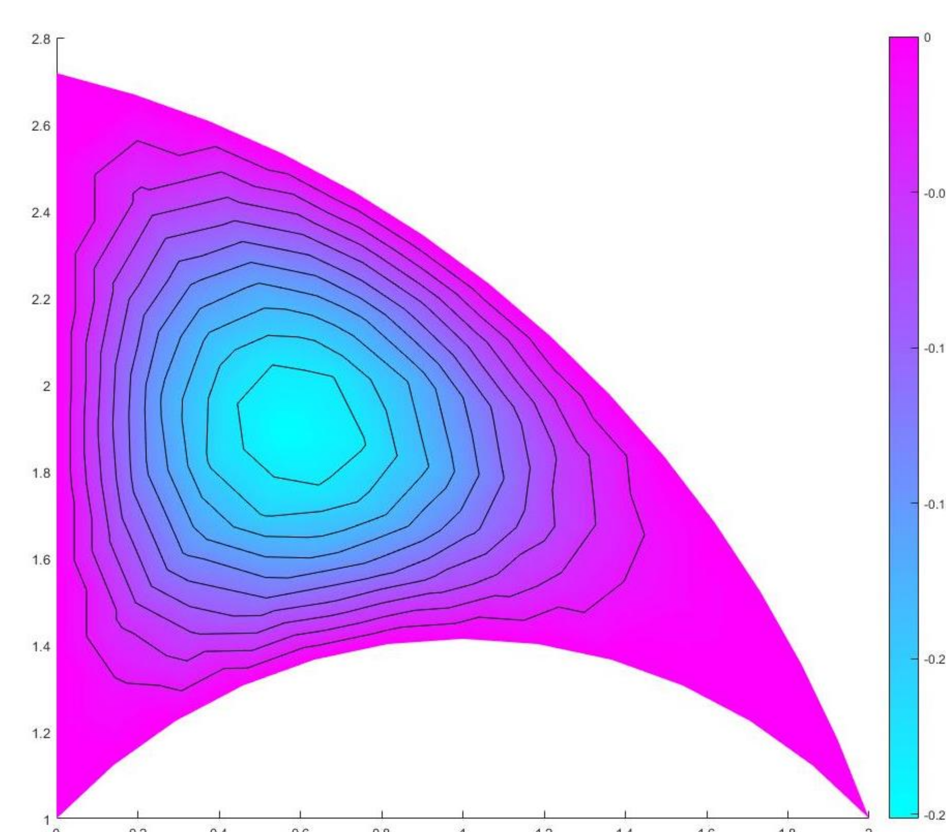
First eigenvalue with respect to x_0 and y_0



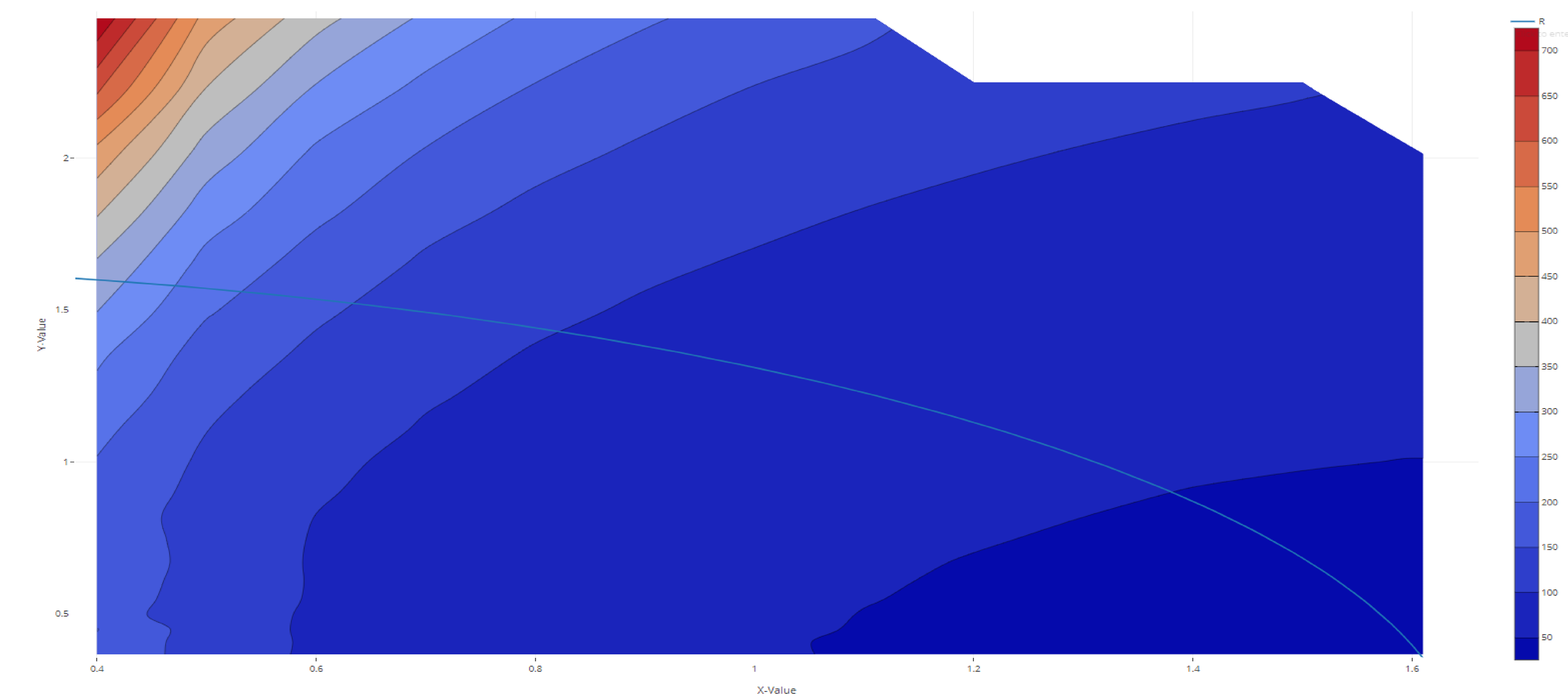
Second eigenvalue with respect to x_0 and y_0



Example of the mesh used *in* finite element method



Graph of the first eigenfunction



Contour diagram of the first eigenvalue with respect to x_0 and y_0

Using Green's theorem and the Rayleigh Quotient, we have the eigenvalues λ_1 of the Laplacian is the minimum value of $R(f)$ among all f satisfying the boundary conditions.

$$R(f) = \frac{\int\int \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \right) dxdy}{\int\int \left(\frac{1}{y^2} f^2\right) dxdy}.$$

Let $C(x_0 + \Delta x, y_0 + \Delta y)$ be the third vertex of the hyperbolic triange whose other vertices are $(0, 1)$ and $(0, e)$. We use Taylor series to estimate the change in the Rayleigh quotient with respect to Δx and Δy in order to obtain a differential equation for the level curve of λ_1 as a function of (x_0, y_0).

Selberg Conjecture & Fundamental Gap Conjecture

1. *Selberg Conjecture:* Consider the group $\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Z}, ad - bc = 1, a \equiv b \equiv 1, b \equiv c \equiv 0 \text{ mod N} \right\}$. $\Gamma(N)$ acts on $\mathbb{H}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}$. Let $X(N)$ be the space of bounded functions f on $\Gamma(N)\backslash\mathbb{H}$. Equivalently, they are the bounded, $\Gamma(N)$- periodic functions on $\mathbb{H}$ We define $\lambda_n(X(N))$ being the n^{th} smallest eigenvalue for the Laplacian on $X(N)$. Then, there is a lower bound for the first non-zero eigenvalue $\lambda_1(X(N))$ such that for $N \geq 1$

$$\lambda_1(X(N)) \geq \frac{1}{4}.$$

2. *Fundamental Gap Conjecture:* Consider the Laplacian on a bounded convex domain Ω in $\mathbb{R}^n$ with Dirichlet boundary conditions. Given the eigenvalues listed in increasing order $0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq ... \to \infty$. Then the difference between the first two eigenvalues satisfies

$$\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{d^2}$$

with d being the diameter of the convex domain Ω.