Double Structures On Conics In $\mathbb{P}^3$

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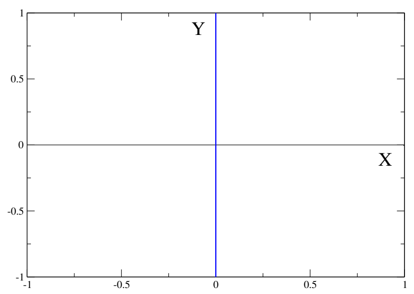
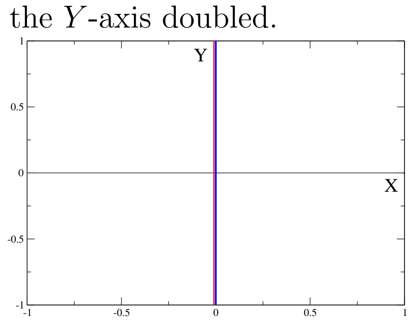
1. Algebraic Curves

Definition 1.1. An algebraic set is the solution set of a system of polynomial equations. The coefficients of the polynomials can be taken from the set real numbers $\mathbb{R}$, or from the set of complex numbers $\mathbb{C}$, or from any field $\mathbb{F}$. An algebraic curve is an one dimensional algebraic set.

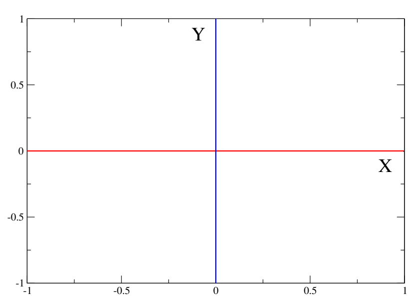
Example 1.2. The solution set of the polynomial equation $x = 0$ in $\mathbb{R}^2$ is the Y-axis.



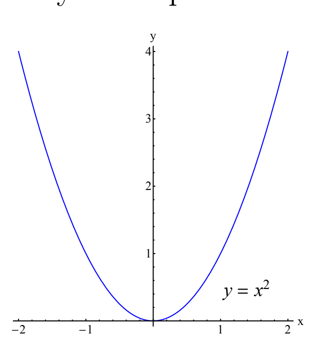
Example 1.3. The solution set of the polynomial equation $x^2 = 0$ in $\mathbb{R}^2$ is the Y-axis doubled.



Example 1.4. The solution set of the polynomial equation $xy = 0$ in $\mathbb{R}^2$ is the union of X and Y-axes.



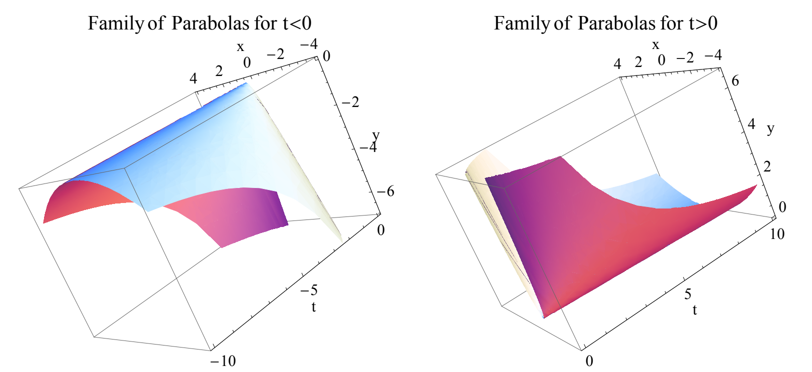
Example 1.5. The solution set of the polynomial equation $y - x^2 = 0$ in $\mathbb{R}^2$ is a parabola whose vertex is the origin and whose axis of symmetry is the positive Y-axis.



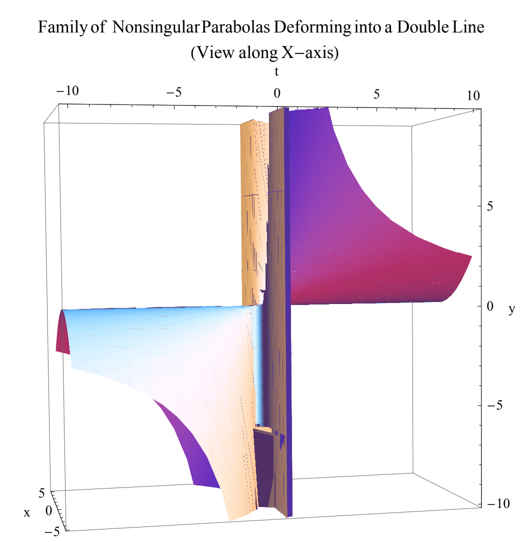
Remark 1.6. Each example above is an algebraic curve. There is a crucial difference between Ex 1.2 and 1.3. Although they have the same set of points, if we count multiplicities each point in Ex 1.3 comes twice. That's why we call it the Y-axis doubled. This is the simplest example of a multiplicity structure on a line.

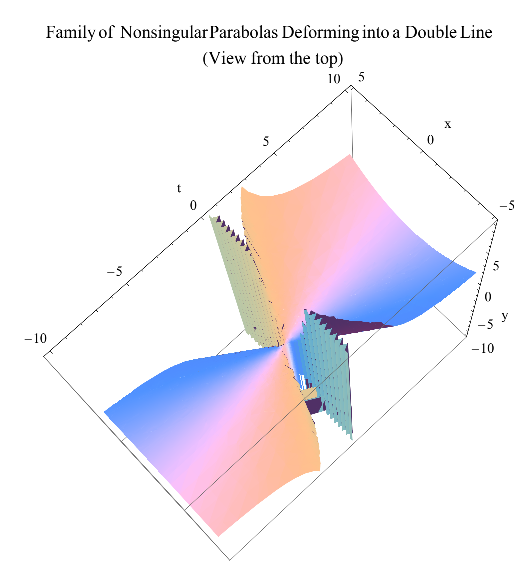
2. Family of Parabolas Deforming into a Double Line

Historically people studied algebraic curves and relations among them very carefully. It turns out that instead of studying them individually it is more fruitful to study them in a 'family'. For example, we can consider the polynomial equation $ty - x^2 = 0$. For each $t \neq 0$ we get a parabola in $\mathbb{R}^3$. If $t = 1$ we get $y = x^2$ (as shown in Ex 1.5), if $t = \frac{1}{2}$ we get $y = 2x^2$ and so on. So this is a family of nonsingular curves, i.e., parabolas.



But for $t = 0$ the equation becomes $x^2 = 0$, whose solution set is the Y-axis doubled. We say that the family of nonsingular parabolas $ty - x^2 = 0$ deforms into the double line $x^2 = 0$ as $t \to 0$.





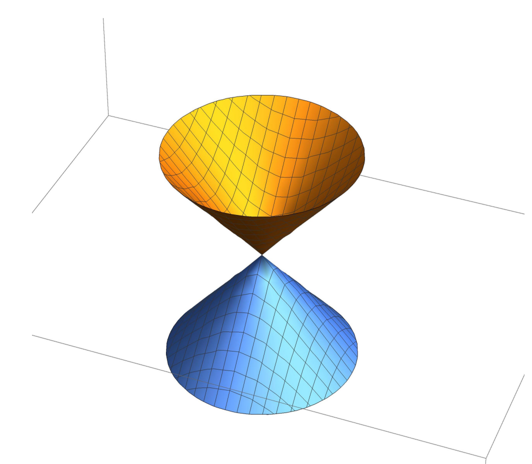
3. Multiplicity Structures

Definition 3.1. Let Y be an algebraic curve. Then a multiplicity structure on Y is another curve Z, which as a set has the same points as Y but with a higher and fixed multiplicity at each point. If the multiplicity is 2, we call Z a double structure on Y.

Example 3.2. The solution set of the polynomial equation $x^2 = 0$ gives a double structure on the Y-axis in $\mathbb{R}^2$, as we have seen in Ex 1.3.

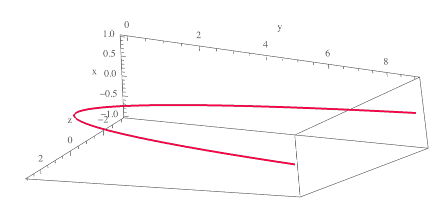
4. Conics

Definition 4.1. A conic is a non-degenerate plane section of a quadric cone.

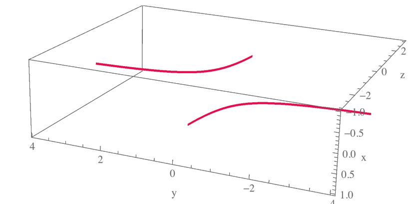


Quadric cone in $\mathbb{R}^3$

Non-degenerate plane sections of this cone give conics, such as parabolas or hyperbolas in the space.



A parabola in $\mathbb{R}^3$



A hyperbola in $\mathbb{R}^3$

Question 4.2. How do double structures on conic look?

In the rest of this poster we will give total descriptions of double conics. We will describe their ideals in $\mathbb{P}^3$ over an algebraically closed field k, which is a more general set up than $\mathbb{R}^3$. Let S be the polynomial ring $k[x, y, z, w]$ with usual grading. Then the quadric cone is the solution set of the polynomial equation $yz - w^2 = 0$. To get a conic we need to intersect this with a plane which is given by the solution set of the equation $x = 0$.

5. Double Conics

Theorem 5.1 (Rabby)**.** Let Z be a double conic on C corresponding to the line bundle $\mathcal{O}_{\mathbb{P}^1}(a)$, where $I_C = (x, q)$ and $q = yz - w^2$.

(1) If a is even then

$$I_Z = (I_C^2, fq - gx),$$

where f, g are homogeneous polynomials in S of degrees $(a+2)/2$ and $(a+4)/2$ respectively, such that their images $\bar{f}, \bar{g}$ in S_C form a regular sequence.

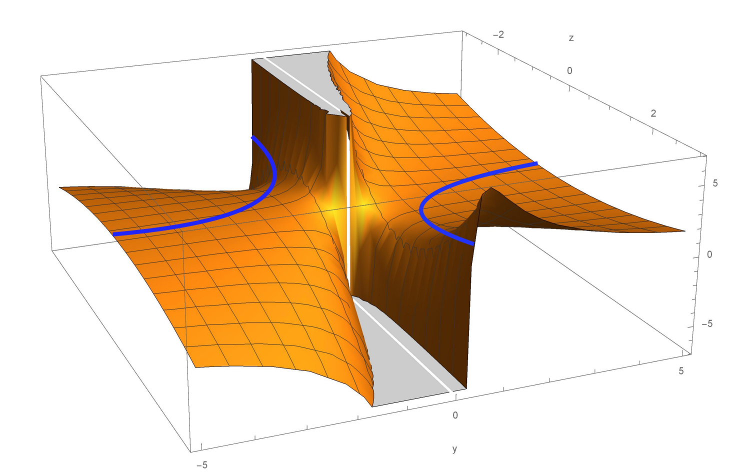
(2) If a is odd then

$$I_Z = (I_C^2, F_1 q - G_1 x, F_2 q - G_2 x),$$

where F_i, G_i are homogeneous polynomials in S of degrees $(a+3)/2$ and $(a+5)/2$ respectively, such that $\{F_1, G_1\}$ and $\{F_2, G_2\}$ are regular sequences in S and $F_1 G_2 - F_2 G_1 \in I_C$.

Proposition 5.2 (Rabby)**.** Let Z be as above. Then
1. $p_a(Z) = -1 - a$,
2. Z is not ACM,
3. proj dim $S_Z = 3$ and
4. Z lies on a surface that is nonsingular along C if and only if a is even.

Example 5.3. Let $a = 0$. Then $\deg f = 1$ and $degg = 2$. Let $f = z, g = y^2$. Notice f and g intersects on a smooth curve in $\mathbb{P}^3$. Then the ideal $I_Z = (I_C^2, zq - y^2 x)$ defines a double structure Z on C. Notice that I_C^2 defines a triple structure on C and the surface $zq - y^2 x = 0$, which is nonsingular along C, cuts out the double structure Z on C. For a odd, such a surface doesn't exist.



Graph of $zq - y^2 x = 0$ *in* $\mathbb{R}^3$.

The next proposition describes the family of all double conics in $\mathbb{P}^3$ corresponding to a given line bundle $\mathcal{O}_{\mathbb{P}^1}(a)$ on C.

Proposition 5.4 (Rabby)**.** Let $\mathcal{H}_Z^a$ be the Hilbert scheme of double conics of type a. Then $\mathcal{H}_Z^a$ is irreducible of dimesion $2a + 16$.

My current research is on the following problem.

Problem 5.5. *Classify the triple conics in* $\mathbb{P}^3$.