Invariants of Triple Conics in $\mathbb{P}^3$

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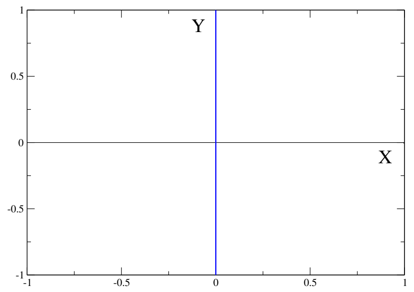
1. Multiplicity Structures on Algebraic Curves

Definition 1.1. An *algebraic set* is the solution set of a system of polynomial equations. The coefficients of the polynomials can be taken from $\mathbb{R}$ or $\mathbb{C}$, or from any field $\mathbb{F}$.

Definition 1.2. An *algebraic curve* is a one dimensional algebraic set.

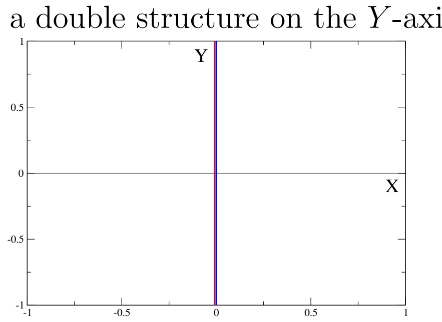
Example 1.3. The solution set of the polynomial equation $x = 0$ in $\mathbb{R}^2$ is the Y-axis.



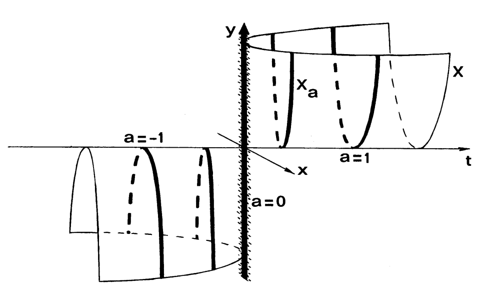
Definition 1.4. Let $X \subset \mathbb{P}^3$ be a nonsingular connected curve. A multiplicity structure on X is some curve Y such that $\operatorname{Supp} Y = \operatorname{Supp} X$. If Y has no embedded or isolated points then its multiplicity can be defined to be the integer

$$\operatorname{mult}(Y) = \frac{\deg Y}{\deg X}.$$

Example 1.5. The solution set of the polynomial equation $x^2 = 0$ in $\mathbb{R}^2$ is a double structure on the Y-axis.



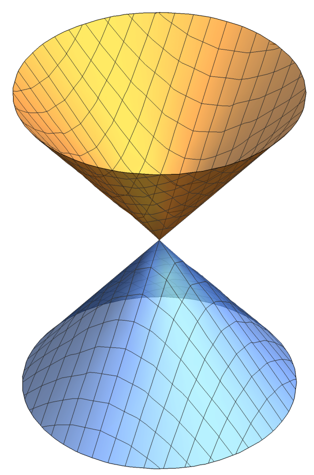
Example 1.6. Consider the polynomial equation $ty - x^2 = 0$. For each $t \neq 0$ we get a parabola in $\mathbb{R}^3$. But for $t = 0$ the equation becomes $x^2 = 0$, whose solution set is a double structure on the Y-axis.



A family of smooth parabolas deforms into a double line.

2. Double Conics

Definition 2.1. A *conic* in $\mathbb{P}^3$ is a degree two integral curve. In other words, every conic in $\mathbb{P}^3$ is a nondegenerate plane section of the quadric cone.



Quadric cone in $\mathbb{R}^3$

Let $\mathbb{P}^3 = \operatorname{Proj} S$, where $S = k[x, y, z, w]$ and k is algebraically closed. Let C be a conic in $\mathbb{P}^3$. Then up to a change of coordinate $I_C = (x, q)$, where $q = yz - w^2$.

Theorem 2.2 (R-). Let Z be a double conic on C with arithmetic genus $-1-\ell$, where $\ell \geq -1$.
(1) If ℓ is even, say $\ell = 2a$, then

$$I_Z = (I_C^2, fq - gx),$$

where f, g are homogeneous polynomials in S of degrees $a+1$ and $a+2$ respectively, such that their images $\bar{f}, \bar{g}$ in S_C form a regular sequence.
(2) If ℓ is odd, say $\ell = 2a+1$, then

$$I_Z = (I_C^2, F_1q - G_1x, F_2q - G_2x),$$

where $\{F_1, G_1\}, \{F_2, G_2\}$ is an admissible pair of sequences of type I on C such that $\deg F_i = a+2$ and $\deg G_i = a+3$.

Remark 2.3. A double conic Z on C of arithmetic genus $-1-\ell$ is a complete intersection if and only if $\ell = -4$ or -2.

Proposition 2.4 (R-). Let Z be a double conic in $\mathbb{P}^3$ of arithmetic genus $-1-\ell$, where $\ell \geq -1$. Then S_Z has projective dimension 3. In particular, Z is not projectively normal.

Theorem 2.5 (R-). A double conic in $\mathbb{P}^3$ is self-linked by complete intersection curves if and only if char $k = 2$.

Proposition 2.6 (R-). Let $\mathcal{H}_Z^\ell$ be the Hilbert scheme of double conics in $\mathbb{P}^3$ of arithmetic genus $-1-\ell$. Then $\mathcal{H}_Z^\ell$ is irreducible of dimesion $2\ell + 16$.

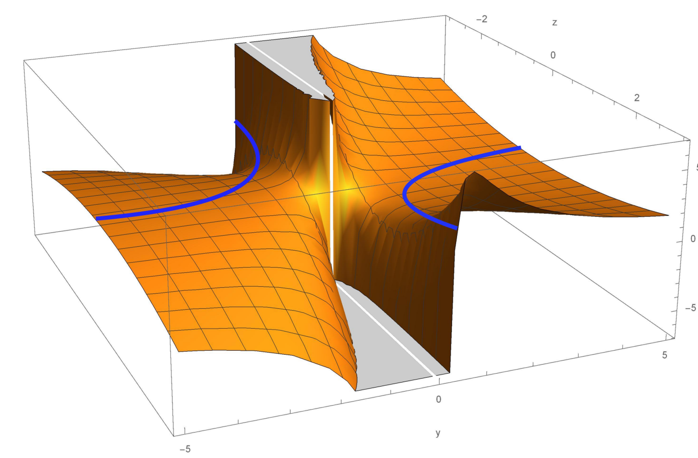
3. Surfaces containing Double Conics

Theorem 3.1 (R-). Let Z be a double conic on C of arithmetic genus $-1-\ell$. If Z is contained in some nonsingular surface F of degree $d > 0$, then $\ell = 2d - 6$.

Proposition 3.2 (R-). Let Z be a double conic on C of arithmetic genus $-1-\ell$. If char $k = 0$ and $d \gg 0$ then a general surface F of degree d containing Z is integral and normal. Moreover, $\operatorname{Sing} F$ is a finite set and

$$|\operatorname{Sing} F| = \begin{cases} 2d - \ell - 6, & \text{if } \ell \text{ is even} \\ 2d - \ell - 4, & \text{if } \ell \text{ is odd.} \end{cases}$$

Example 3.3. Let $\ell = 0$. Then $I_Z = (I_C^2, zq - y^2x)$ defines a double conic Z on C. Notice, I_C^2 defines a triple structure on C and the surface $zq - y^2x = 0$, which is nonsingular along C, cuts out the double conic Z.



Graph of $zq - y^2x = 0$ *in* $\mathbb{R}^3$.

4. Quasi-primitive and Thick Extensions

Definition 4.1. Let $X \subset \mathbb{P}^3$ be a nonsingular connected curve and let Y be a multiplicity structure on X.
(1) Y is called a *quasi-primitive* extension of X if it has embedding dimension two at all but finitely many points.
(2) Y is called the *thick* extension of X if it has embedding dimension three at each point. Also in that case $I_Y = I_X^2$.

Remark 4.2. If Y is a multiplicity structure on a nonsingular connected curve $X \subset \mathbb{P}^3$ then it is either a quasi-primitive or a thick extension.

Example 4.3. Let $X \subset \mathbb{P}^3$ be the line given by $I_X = (x, y)$. Then $I_Y = (x^2, xy, y^3, y^2z - w^2x)$ defines a quasi-primitive triple line Y on X. On the other hand, $I_X^2 = (x^2, xy, y^2)$ defines the thick triple line on X.

5. Structure Theorem of Triple Conics

Theorem 5.1 (R-). Let Z be a double conic on C of type ℓ with $I_Z \otimes S_C \subseteq S_C[m] \oplus S_C[n]$. Let $\phi : I_Z \to S_C[2\ell + c]$ be the map defined as

$$\phi : I_Z \to I_Z \otimes S_C \subseteq S_C[m] \oplus S_C[n] \xrightarrow{\psi} S_C[2\ell + c],$$

where $c \in \mathbb{Z}_{\geq 0}$ and $\operatorname{Coker} \psi$ has finite length. Then $\operatorname{Ker} \phi$ is the total ideal of a CM triple conic W on C of type (ℓ, c), having Z as the second CM filtrant. Moreover,

$$I_W = \operatorname{Ker} \phi = I_C I_Z + j^{-1} \operatorname{Ker}(\tau),$$

where j is the inclusion of $I_Z \otimes S_C$ in $T(m) \oplus T(n)$ and τ is the map corresponding to ψ.

6. Invariants of Triple Conics

Theorem 6.1 (R-). Let W be a quasi-primitive triple conic on C with Z as the second CM filtrant.
(1) If W has type $(2a, 2b)$ then

$$I_W = (I_C I_Z, \alpha x^2 + \beta xq + \gamma q^2 - R(fq - gx)),$$

where $a \geq 0$ and $b \geq 2$.
(2) If W has type $(2a, 2b+1)$ then

$$I_W = (I_C I_Z, H_1 - R_1(fq - gx), H_2 - R_2(fq - gx)),$$

where $a \geq 0$ and $b \geq 1$.
(3) If W has type $(2a+1, 2b)$ then

$$I_W = (I_C I_Z, H_1 - R(F_1q - G_1x), H_2 - R(F_2q - G_2x)),$$

where $a \geq -1$ and $b \geq 1$.
(4) If W has type $(2a+1, 2b+1)$ then

$$I_W = (I_C I_Z, H - R_1(F_1q - G_1x) - R_2(F_2q - G_2x)),$$

where $a \geq -1$ and $b \geq 1$

Example 6.2. Let $I_W = (x^3, q)$. Then W is a triple conic in $\mathbb{P}^3$. Moreover, W is a complete intersection.