Deeper Exploration of the C*-Algebras Arising from Uniformly Recurrent Subgroups and their Relationship with Crossed Products

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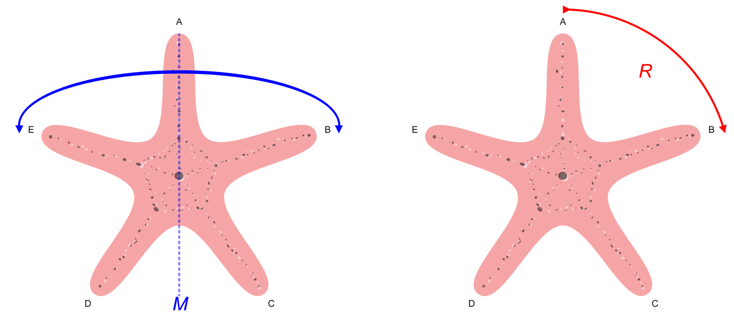
Groups

When an object is symmetrical, it may be transformed in a way that preserves its appearance. If we combine two such transformations, we end up finding a new symmetry.

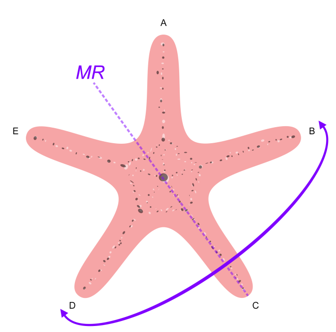
A **group** can be thought of as representing the structure of the symmetries of an object: it is a set containing all these transformations, as well as information about what results from their combination.

Example: D_{10}, the dihedral group of ten elements.

This represents the symmetries of a regular pentagon, as well as many other five-sided shapes. The group is generated by two fundamental transformations: M mirrors the image along the vertical axis and R rotates it $2\pi/5$ radians.

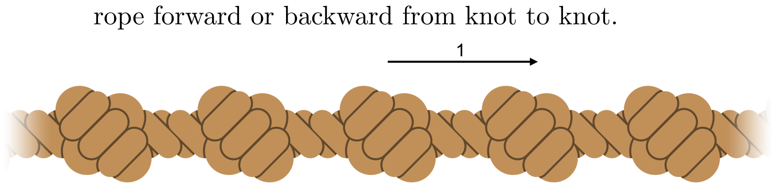


If we combine these two transformations by rotating then mirroring, denoted MR, we find a new symmetry. Through this method we can find them all.



Example: $\mathbb{Z}$, the integers.

Groups can also be infinite. Consider a taut rope of infinite length, with knots at regular increments. Its symmetries form this well-known group, with transformations of moving the rope forward or backward from knot to knot.



Example: $\mathbb{F}_2$, the free group on two generators.

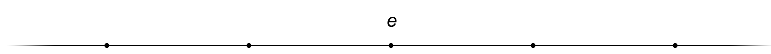
Groups can also be much more abstract. This group comprises all finite strings of two symbols and those symbols' inverses. To conceptualize this as representing symmetries, one may need to broaden one's sense of "objects" and "transformations."

Cayley graphs

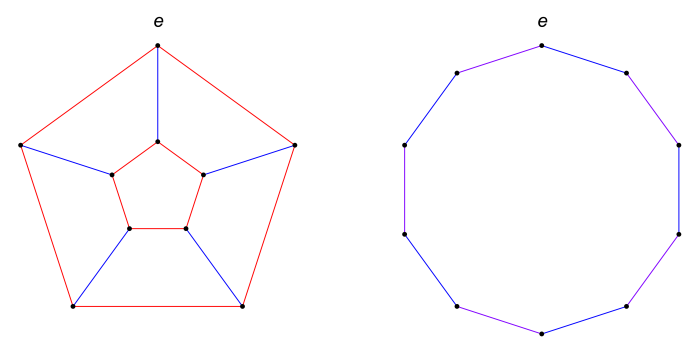
The examples of groups given here have all been *finitely generated*: there is a finite list of fundamental transformations which we can combine to generate all possible symmetries. This suggests a novel method of "drawing pictures" of our groups, first considered by Arthur Cayley in 1878, hence called a **Cayley graph**.

Our method consists of drawing vertices to represent each of the symmetries, with edges between them representing the fundamental transformations. (Different transformations can be illustrated using different colors) Any path along these edges starting at the point we call e represents a combination of transformations that will yield the symmetry at the end.

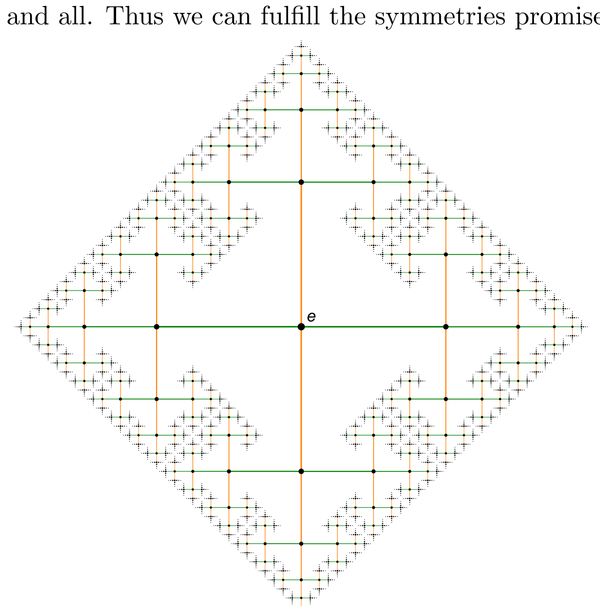
Example: Our rope with knots is just a 3D form of the Cayley graph of $\mathbb{Z}$.



Example: Using M & R gives us the Cayley graph of D_{10} as on the left. However, Cayley graphs can depend on the choice of generators, as depicted on the right where we instead use M & MR to generate D_{10}.



Example: We can also go the other way and create groups from graphs. Symmetries of the graph replace e with another vertex and rearrange the rest so that it looks the same, colors and all. Thus we can fulfill the symmetries promised by $\mathbb{F}_2$.



C*-algebras

C*-algebras generalize two mathematical structures, relating them to the broader subjects of noncommutative geometry and operator algebras:

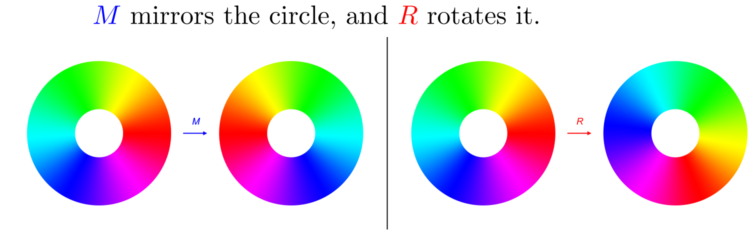
- $C(X, \mathbb{C})$, the complex-valued continuous functions on a compact space. From topology. Can represent the temperature at every point of a room. Uses the uniform norm, which measures the greatest magnitude a function attains. Is not finite-dimensional.
- $\mathbb{M}_n(\mathbb{C})$, the $n \times n$ matrices with complex-valued entries. From linear algebra. Represents the linear operators on an n-dimensional vector space. Uses the operator norm, which can be measured from eigenvalues. Is not commutative.

Group actions

The relationships between the symmetries in a group can be expressed onto other objects—including C*-algebras—via what is known as a **group action**.

Example: D_{10} acts on the unit circle $\mathbb{T} := \{e^{2\pi i x} \mid x \in \mathbb{R}\}$.

This action perfectly mimics the symmetries of a pentagon: M mirrors the circle, and R rotates it.



An action on a compact space induces an action on the C*-algebra of continuous functions by shifting to the argument, so we have that D_{10} acts on $C(\mathbb{T}, \mathbb{C})$ via:

$$(Mf)(e^{2\pi i x}) = f(e^{-2\pi i x}), \qquad (Rf)(e^{2\pi i x}) = f(e^{2\pi i (x+1/5)}).$$

Example: D_{10} acts on the 5D complex vector space $\mathbb{C}^5$.

This representation permutes a vector's coordinates in the same way the pentagon's corners are permuted:

$$M \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} A \\ E \\ D \\ C \\ B \end{bmatrix}, \qquad R \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} E \\ A \\ B \\ C \\ D \end{bmatrix}.$$

Thus the transformations become matrices:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Once again, this induces an action on $\mathbb{M}_5(\mathbb{C})$ via inner automorphisms, where we multiply matrices by the group element on the left and its inverse on the right.

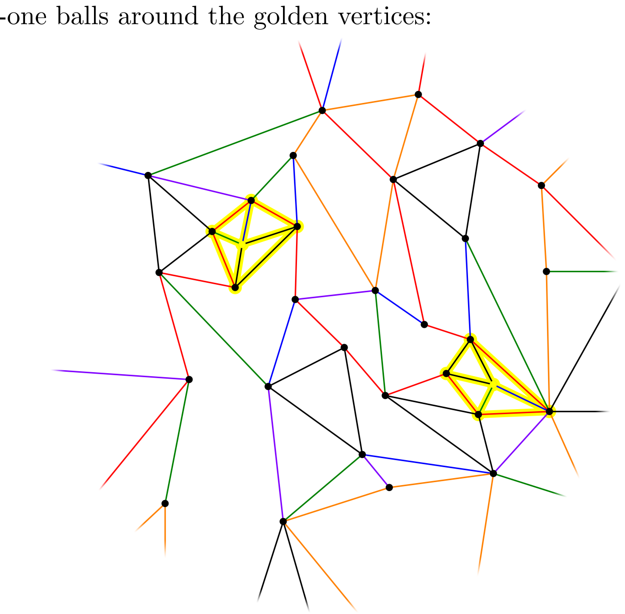
Constructions and crossed products

Given a C*-algebra $\mathcal{A}$, there are several ways to construct new ones, the most basic being $C(X, \mathcal{A})$ & $\mathbb{M}_n(\mathcal{A})$, which are respectively just continuous functions/matrices that have values/entries in $\mathcal{A}$ instead of $\mathbb{C}$.

If $\mathcal{A}$ is acted on by a group Γ, it forms a dynamical system from which we construct a new C*-algebra known as the **crossed product**. In truth, different norms can produce many different crossed products built from the same dynamical system. However, only two are commonly encountered: the "reduced" crossed product $\mathcal{A} \rtimes_r \Gamma$, which is the most naturally-occurring form; and the "full" crossed product $\mathcal{A} \rtimes \Gamma$, which is the largest of them all.

C*-algebras of URS's

Gábor Elek introduced a new way of constructing a C*-algebra $C_r^*(\mathcal{U})$ from a Schreier graph (a generalization of a Cayley graph) of a *Uniformly Recurrent Subgroup* (URS) $\mathcal{U}$. He used matrices with entries indexed by the graph's vertices, with the requirement that entries be similar when the subgraphs containing the corresponding vertices look similar. To illustrate, the highlighted subgraphs below are isomorphic radius-one balls around the golden vertices:



New results

Theorem 1: The C*-algebra of a URS is a crossed product under a certain norm $\|\cdot\|_{\mathcal{O}}$:

$$C_r^*(\mathcal{U}) = C(\mathcal{U}) \rtimes_{\mathcal{O}} \Gamma.$$

This crossed product is not, in general, the full *or* reduced crossed product.

Theorem 2: $\mathcal{U}$ is generic iff $C(\mathcal{U})$ is a *C*-diagonal subalgebra* of $C_r^*(\mathcal{U})$.