Metrics of Positive Scalar Curvature

on Riemannian Manifolds

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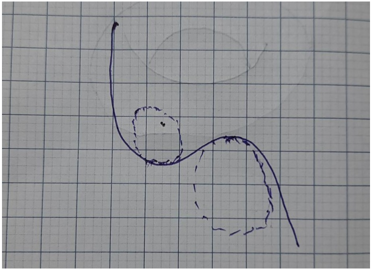
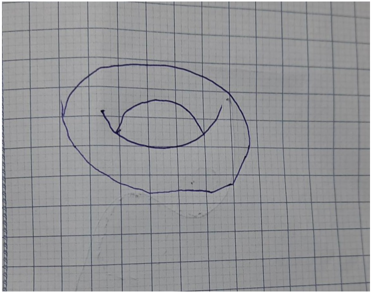
Intuitive concept of curvature





The metric tensor on manifolds

- Let M be an n-dimensional smooth manifold and $p \in M$. A symmetric nondegenerate bilinear form assigning at p an inner product g_p on the tangent space $T_p(M)$ is called a metric tensor on M.
- Denote $\partial_1, \partial_2, \ldots, \partial_n$ the n coordinate vector fields of an n-dimensional manifold. Then the metric tensor components g_{ij} would be

$$g_{ij} = \langle \partial_i, \partial_j \rangle$$

- The inverse metric is denoted g^{ij}, so $\sum_k g_{ik} g^{kj} = \delta_{ij}$
- Intuitively, it's a way to measure distance on curved space (analogy: the Pythagorean theorem in flat space).

Some useful formulas in Riemannian geometry

- The Christoffel symbols with upper index i and lower indices j and k:

$$\Gamma^i_{jk} = \sum_m g^{im} \quad \frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} \quad \frac{\partial g_{jk}}{\partial x^m}$$

- The components of the Riemann tensor (with i as upper index and j,k,l lower indices):

$$R^i_{jkl} = \frac{\partial}{\partial x^l}\Gamma^i_{kj} \quad \frac{\partial}{\partial x^k}\Gamma^i_{lj} + \sum_m \Gamma^i_{lm}\Gamma^m_{kj} \quad \sum_m \Gamma^i_{km}\Gamma^m_{lj}$$

- The scalar curvature of M:

$$S = \sum_{i,j,k} g^{ij} R^k_{ijk}$$

Scalar curvature and volumes of balls

Scalar curvature and volumes of balls

Theorem (Known result)

Let M be an n-dimensional Riemannian manifold and $a\ 2\ M$. Consider the geodesic ball $B_r(a)$ centered at a, of radius r (r is small). Then,

$$V(B_r(a)) = V_f \quad 1 \quad \frac{r^2}{6(n+2)} S(a) + O(r^3))$$

where V_f denotes the volume of a ball of the same radius but in R^n and S is the scalar curvature of M at p.

Part of the proof of the theorem uses the following lemma

Lemma (Known result)

Let a be a point in an n-dimensional Riemannian manifold M. Then, the Taylor expansion of the metric tensor in a geodesic normal neighborhood of a is:

$$g_{ij}(x) = {}_{ij} + \mathrm{X} \frac{1}{3} x^p x^q \ {}_{iu} R^u_{pqj}(a) + O(jxj^3)$$

Scalar curvature and volumes of balls

Why do we care above PSC (Positive Scalar Curvature Metrics?

- It is commonly known in the research community that on a manifold of dimension 3 and higher, we can always obtain a metric so that its scalar curvature is negative.
- It is an open question about whether or not there is a PSC metric on certain kinds of manifolds of dimension 3 or higher.

Product Manifolds and PSC Metrics

Lemma (Known result)

Let M and N be Riemannian manifolds with metrics g_1 and g_2 and scalar curvature S_1 and S_2 respectively. Then the scalar curvature of $M \times N$, the product of M and N, will be $S = S_1 + S_2$.

Lemma (Known result)

Let M be a Riemannian manifold with metric g, and let $t \in \mathbb{R}^+$ be arbitrary. Denote S the scalar curvature of M with respect to g. Then, tg is another metric on M, and the scalar curvature S' of M with respect to tg will be $S' = \frac{1}{t} S$

Product Manifolds and PSC Metrics

Now, we will come to the main theorem that I found out during my research

Theorem (Known result)

Let M and N be compact Riemannian manifolds. Suppose further that M admits a metric of positive scalar curvature (PSC). Then, there exists a metric on M N that is a PSC metric.

Future research objective

- To investigate manifolds where there is a product-like structure at every point and see whether or not there is a PSC metric structure on the manifold.

Thank You