Metrics of Positive Scalar Curvature on Riemannian Manifolds

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Khoi Nguyen (Texas Christian University) Metrics of Positive Scalar Curvature

Intuitive concept of curvature





- Let M be an n-dimensional smooth manifold and p ∈ M. A symmetric nondegenerate bilinear form assigning at p an inner product g_p on the tangent space T_p(M) is called a metric tensor on M.
- Denote $\partial_1, \partial_2, \cdots, \partial_n$ the n coordinate vector fields of an n-dimensional manifold. Then the metric tensor components g_{ij} would be

$$g_{ij} = \langle \partial_i, \partial_j \rangle$$

- The inverse metric is denoted g^{ij} , so $\sum_{k} g_{ik}g^{kj} = \delta_{ij}$
- Intuitively, it's a way to measure distance on curved space (analogy: the Pythagorean theorem in flat space).

Some useful formulas in Riemannian geometry

• The Christoffel symbols with upper index i and lower indices j and k:

$$\Gamma_{jk}^{i} = \sum_{m} g^{im} \left(\frac{\partial g_{jm}}{\partial x^{k}} + \frac{\partial g_{km}}{\partial x^{j}} - \frac{\partial g_{jk}}{\partial x^{m}} \right)$$

• The components of the Riemann tensor (with i as upper index and j,k,l lower indices):

$$R_{jkl}^{i} = \frac{\partial}{\partial x^{l}} \Gamma_{kj}^{i} - \frac{\partial}{\partial x^{k}} \Gamma_{lj}^{i} + \sum_{m} \Gamma_{lm}^{i} \Gamma_{kj}^{m} - \sum_{m} \Gamma_{km}^{i} \Gamma_{lj}^{m}$$

• The scalar curvature of M:

$$S = \sum_{i,j,k} g^{ij} R^k_{ijk}$$

Scalar curvature and volumes of balls



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Theorem (Known result)

Let M be an n-dimensional Riemannian manifold and $a \in M$. Consider the geodesic ball $B_r(a)$ centered at a, of radius r (r is small). Then,

$$V(B_r(a)) = V_f\left(1 - rac{r^2}{6(n+2)}S(a) + O(r^3)
ight)$$

where V_f denotes the volume of a ball of the same radius but in \mathbb{R}^n and S is the scalar curvature of M at p.

Part of the proof of the theorem uses the following lemma

Lemma (Known result)

Let a be a point in an n-dimensional Riemannian manifold M. Then, the Taylor expansion of the metric tensor in a geodesic normal neighborhood of a is:

$$g_{ij}(x) = \delta_{ij} + \sum \frac{1}{3} x^{p} x^{q} \delta_{iu} R^{u}_{pqj}(a) + O(|x|^{3})$$

Scalar curvature and volumes of balls



- It is commonly known in the research community that on a manifold of dimension 3 and higher, we can always obtain a metric so that its scalar curvature is negative.
- It is an open question about whether or not there is a PSC metric on certain kinds of manifolds of dimension 3 or higher.

Lemma (Known result)

Let M and N be Riemannian manifolds with metrics g_1 and g_2 and scalar curvature S_1 and S_2 respectively. Then the scalar curvature of $M \times N$, the product of M and N, will be $S = S_1 + S_2$.

Lemma (Known result)

Let *M* be a Riemannian manifold with metric *g*, and let $t \in \mathbb{R}^+$ be arbitrary. Denote *S* the scalar curvature of *M* with respect to *g*. Then, t*g* is another metric on *M*, and the scalar curvature *S*' of *M* with respect to t*g* will be $S' = \frac{1}{t}S$ • Now, we will come to the main theorem that I found out during my research

Theorem (Known result)

Let M and N be compact Riemannian manifolds. Suppose further that M admits a metric of positive scalar curvature (PSC). Then, there exists a metric on $M \times N$ that is a PSC metric.

• To investigate manifolds where there is a product-like structure at every point and see whether or not there is a PSC metric structure on the manifold.

Thank You

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