Group Actions on Cell Complexes

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△ and CW Complexes

In this project, we use a combination of Δ-complexes and CW-complexes to represent spaces we are investigating. Roughly speaking, representing a space as a complex is done by gluing its 0-dimensional, 1-dimensional, and 2-dimensional parts together (this can be extended further into higher dimensions).

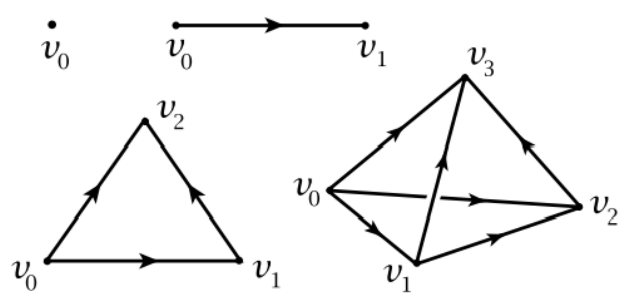

Figure 1. Visual examples of 0,1,2, and 3 simplexes.

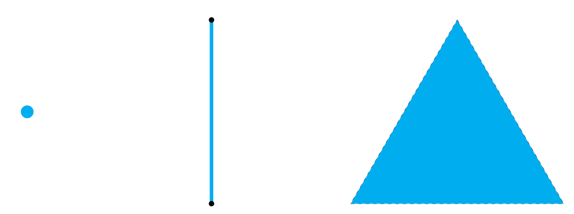

Figure 2. Visual examples of a 0-cell, 1-cell, and 2-cell.

In Figure 1, the singular edge is a simplex and is denoted as $[v_0, v_1]$. Order does matter in this case (so $[v_0, v_1]$ is not the same simplex as $[v_1, v_0]$) and is represented as an arrow in the image. Removing all the lower dimensional parts of a n-simplex gives us the face, or n-cell representation, of that part. So for the 2-simplex $[v_0, v_1, v_2]$ in Figure 1, removing the edges and vertices leaves us with a 2-cell, like depicted on the right of Figure 2.

Example: △ and CW Complexes of a Torus

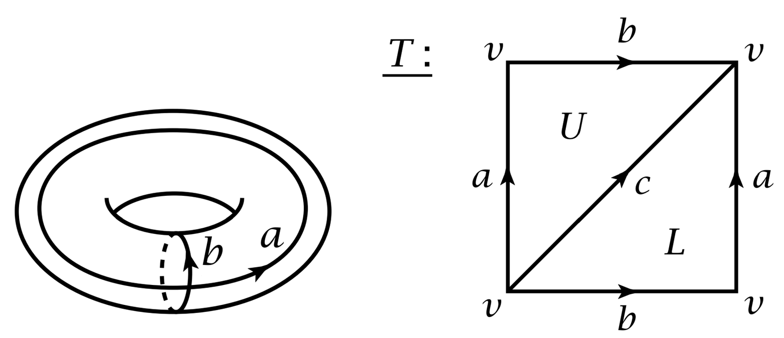

Figure 3. Representations of a Torus

Consider the representations of a torus in Figure 3. The corresponding CW-complex* is $\{v, a, b, c, U, L\}$ where v is a 0-cell, a b and c are 1-cells, and U,V are 2-cells with directions and gluings as identified in the image. The corresponding Δ-complex* is $\{[v_1], [v_2], [v_3], [v_4], [v_3, v_1], [v_4, v_2], [v_1, v_2], [v_3, v_4], [v_3, v_2], [v_1, v_3, v_2], [v_2, v_3, v_4]\}$where the members of each set $\{[v_1], [v_2], [v_3], [v_4]\}$, $\{[v_3, v_1], [v_4, v_2]\}$, and $\{[v_1, v_2], [v_3, v_4]\}$ are glued together. You may see that CW-complexes are better at conveying the image of the torus but Δ-complexes are more rigorous. I will continue primarily using CW-complexes to represent our spaces but will appeal to Δ-complexes when defining the boundary map and doing calculations.

*I should say an intuitive representation of these complexes. The rigorous definitions of CW and Δ complexes do not match here.

Simplicial Homology

The motivation behind utilizing homology is to generate invariants for us to tell different spaces apart from each other. We do this by generating homology groups H_i that depend on boundary map functions.

The Boundary Map

Let X be a Δ-complex. Let C_n denote the free abelian group of open n-simplexes in X with integer coefficients. Then we define a class of functions $\partial_n : C_n \to C_{n-1}$ based off of the basis elements $\sigma = [v_0, \ldots, v_n] \in C_n$ such that

$$\partial_n(\sigma) = \sum_i (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

where the hat over v_i means that the vertex is deleted from the list.

This may seem like a lot to chew at first, but a few examples may help:

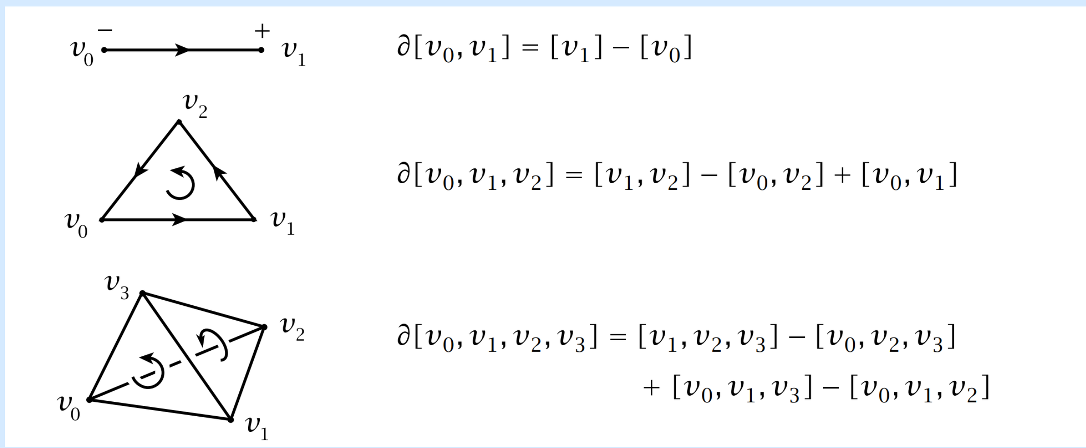

Figure 4. Examples of computing the boundary map.

Homology groups

With the boundary map as defined above, we find that

$$\ldots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

with $\partial_n \circ \partial_{n-1} = 0$. Using this result and a fair amount of imagination we find that the homology groups

$$H_n = \ker \partial_n / \operatorname{Im} \partial_{n+1}$$

of Δ-complexes X and Y are isomorphic if X and Y are topologically equivalent. Then the homology groups H_n are topological invariants we can use to tell different spaces apart!

Example: Homology Groups of a Torus

Here I will compute the homology groups for the torus. Lets use the CW-complex of the torus from Figure 3. Then $\ker \partial_0 = \operatorname{span}\{v\}, \ker \partial_1 = \operatorname{span}\{a, b, c\}$, $\ker \partial_2 = \operatorname{span}\{U + L\}$, $\operatorname{Im} \partial_1 = \operatorname{span}\{0\}, \operatorname{Im} \partial_2 = \operatorname{span}\{a + b - c, c - b - a\}$, $\operatorname{Im} \partial_3 = \operatorname{span}\{0\}$. Then $H_0 = \operatorname{span}\{v\}/\operatorname{span}\{0\} \cong \mathbb{Z}$, $H_1 = \operatorname{span}\{a, b, c\}/\operatorname{span}\{a + b - c\} \cong \mathbb{Z} \oplus \mathbb{Z}$, $H_2 = \operatorname{span}\{U + L\}/\operatorname{span}\{0\} \cong \mathbb{Z}$, and $H_n = 0$ for $n \geq 3$.

Group Actions, Representation Theory, and Homology

A focus of this project is to apply various group actions to cell complexes, use representation theory to generate C_n groups, and use homology to distinguish these groups and group actions from each other.

Group Actions

An action of a group G on a space X is a function $f : X \times G \to X$, written $gx = f(x, g_0)$, such that $ex = x$ for all $x \in X$ and $(g_1 g_2)x = g_1(g_2 x)$ for all $x \in X$ and $g_1, g_2 \in G$. We primarily focus on group actions over finite abelian groups.

Representation Theory

Let G be a group and $\mathbb{F}$ be a field. Then an $\mathbb{F}$ representation of G is an $\mathbb{F}-$vector space V paired with a group homomorphism $\rho : G \to GL(V)$. For our purposes, $\mathbb{F} = \mathbb{C}$. Since $\operatorname{char}(\mathbb{C}) = 0$ and we only consider finite abelian groups, all of our proposed representations are completely reducible to 1-dimensional representations.

Generation of C_n^ρ Group

Let X be a Δ-complex, g be a group action of a cyclic finite abelian group G on X, C_n denote the free abelian group of n-simplexes with complex coefficients, and let $\rho : G \to \mathbb{C}^\times$ be an irreducible complex-valued representation. Let $C_n^\rho = \{c_1 a_1 + \ldots + c_i a_i \mid g(c_1 a_1 + \ldots + c_i a_i) = \rho(g)(c_1 a_1 + \ldots + c_i a_i)\}$ where $c_1, \ldots, c_i$ are arbitrary complex numbers and $a_1, \ldots, a_i$ are the basis elements of C_n.

Generation of H_n^ρ groups

Keep the boundary map ∂_n as defined in the last column. Then the groups H_n^ρ can be defined as

$$H_n^\rho = (\ker \partial_n \cap C_n^\rho)/(\operatorname{Im} \partial_{n+1} \cap C_n^\rho)$$

By similar argument to the regular homology groups, the $H_n^{\rho_k}$ groups are invariant between topologically equivalent spaces and choice of group action representation.

References

[1] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.

[2] Thomas W. Judson. *Abstract Algebra Theory and Applications*. PWS Publishing, 2016.

[3] Mark W. Meckes. A brief introduction to group representations and character theory. Lecture Notes, 2018.

Example Using Homology and Group Action Representations

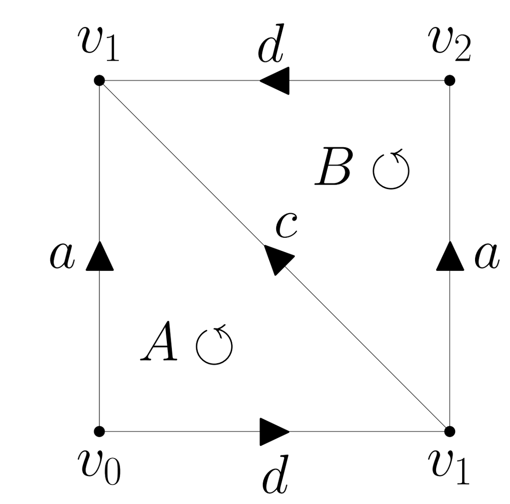

Figure 5.

Let X be the $\Delta-$complex depicted in Figure 5. Let $G = \mathbb{Z}_2 = \{e, q\}$ and let G act on X by rotating $180°$ counter-clockwise. Let the values of this group action be defined in the table below on the group element q. Then

x	qx	$\partial_n(x)$
v_0	v_0	0
a	$-a$	0
c	$-c$	0
d	d	0
A	B	$c + d - a$
B	A	$a + d - c$

$$H_0^{\rho_0} = \frac{\ker \partial_0 \cap C_0^{\rho_0}}{\operatorname{Im} \partial_1 \cap C_0^{\rho_0}} = \frac{\operatorname{span}\{v_0\}}{\{0\}} \cong \mathbb{Z},$$

$$H_1^{\rho_0} = \frac{\ker \partial_1 \cap C_1^{\rho_0}}{\operatorname{Im} \partial_2 \cap C_0^{\rho_0}} = \frac{\operatorname{span}\{d\}}{\operatorname{span}\{-2d\}} \cong \mathbb{Z}_2$$

$$H_2^{\rho_0} = \frac{\ker \partial_2 \cap C_2^{\rho_0}}{\operatorname{Im} \partial_3 \cap C_0^{\rho_0}} \cong \{0\},$$

$$H_0^{\rho_1} = \frac{\ker \partial_0 \cap C_0^{\rho_1}}{\operatorname{Im} \partial_1 \cap C_0^{\rho_1}} \cong \{0\},$$

$$H_1^{\rho_1} = \frac{\ker \partial_1 \cap C_1^{\rho_1}}{\operatorname{Im} \partial_2 \cap C_0^{\rho_1}} \cong \frac{\operatorname{span}\{a, c\}}{\operatorname{span}\{2(a - c)\}} \cong \mathbb{Z} \oplus \mathbb{Z}_2,$$

$$H_2^{\rho_1} = \frac{\ker \partial_2 \cap C_2^{\rho_1}}{\operatorname{Im} \partial_3 \cap C_0^{\rho_1}} \cong \{0\}.$$

Ongoing Goals

Classify all possible triangular Δ-complexes paired with various group actions up to isomorphism.

Write a program to compute the homology of Δ-complexes to aid with the above goal and extend to higher dimensions.

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