An Investigation into Riemannian Manifolds of Positive Scalar Curvature

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Abstract

In the field of differential geometry, the condition on the Riemannian metric so that a manifold has positive scalar curvature (PSC) is important for a number of reasons. In this project, we study more about PSC metrics on such manifolds. Specifically, we refine and provide some details to the proof of Gromov and Lawson that the connected sum of 2 n-dimensional manifolds will admit a PSC metric, provided each of the manifold has a metric with the same condition. We then derive some useful formulas related to the Riemann curvature tensor, the Ricci tensor, and the scalar curvature for a manifold equipped with an adapted local orthonormal frame and its dual coframe. We also compute the relationships between the scalar curvature S of a manifold M and the decomposition into S_1 and S_2 when dividing the tangent bundle into the direct sum of two different subbundles. Our ultimate goal with these computations is to eventually come up with some conditions on the structure of a Riemannian manifold and its corresponding metric in order for it to have PSC.

Preliminary facts

In classical differential geometry, one of the most important notions of the curvature of surfaces is the Gaussian curvature.
Extending the Gaussian curvature to higher-dimensional manifolds, we get an analogous notion of the scalar curvature of a Riemannian manifold.

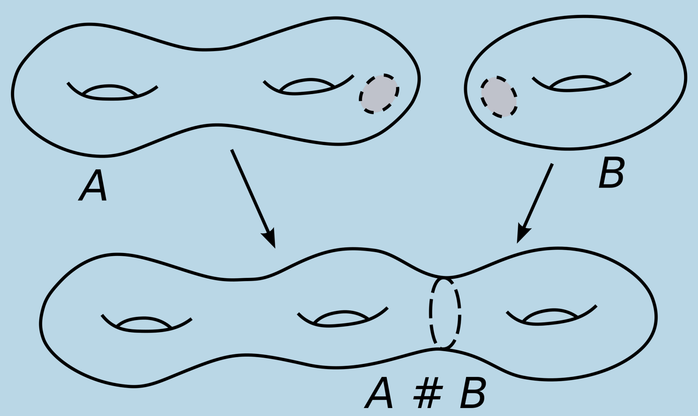
Definition 1. *Let M be an n-dimensional manifold and $p \in M$. A symmetric positive definite bilinear form g assigning at p an inner product g_p on the tangent space $T_p(M)$ is called a **metric tensor** (or **Riemannian metric**) on M. The pair (M, g) is then called a **Riemannian manifold***

From the metric tensor, we can calculate various quantities relating to the scalar curvature of a Riemannian manifold, such as the Riemann tensor, the Ricci tensor, and finally the scalar curvature.

Some known results

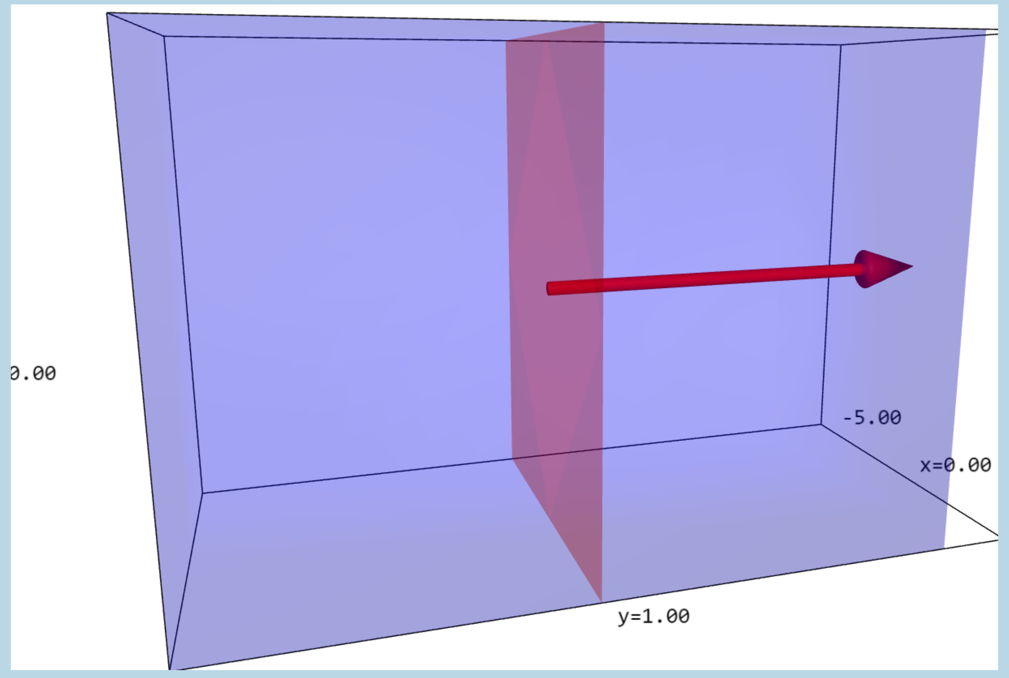
Theorem 1. *Let M and N be compact Riemannian manifolds. Suppose further that M admits a metric of positive scalar curvature (PSC). Then, there exists a metric on $M \times N$ that is a PSC metric.*

Theorem 2 (Gromov-Lawson). *If M and N are compact n dimensional manifolds, with $n \geq 3$, having positive scalar curvature metrics, then their connected sum also has a positive scalar curvature metric.*



The set up of our problem

Consider a n-dimensional Riemannian manifold (M, g) and its tangent bundle TM. Suppose we can split TM into two orthogonal subbundle X and Y, rank r and s respectively. In other words, $TM = X \oplus Y$. We will deform the metric on M by scaling g_Y, the underlying metric restricted to Y, by $\frac{1}{a^2}$, and observe the change in the scalar curvature S according to the said deformation.



Useful constructions

Definition 2. *Let (M, g) be a Riemannian manifold. A set $\beta = \{e_1, e_2, \cdots, e_n\}$ of vector fields defined on a neighborhood U of $p \in M$ is a **local orthonormal frame** if $\{e_1(x), e_2(x), \cdots, e_n(x)\}$ is an orthonormal basis of $T_x(M)$ for all $x \in U$.*

Definition 3. *Let V, W be vector fields on a Riemannian manifold M. The **Lie bracket** of V and W is a vector field on M, which is determined by*

$$[V, W](f) = V(W(f)) - W(V(f))$$

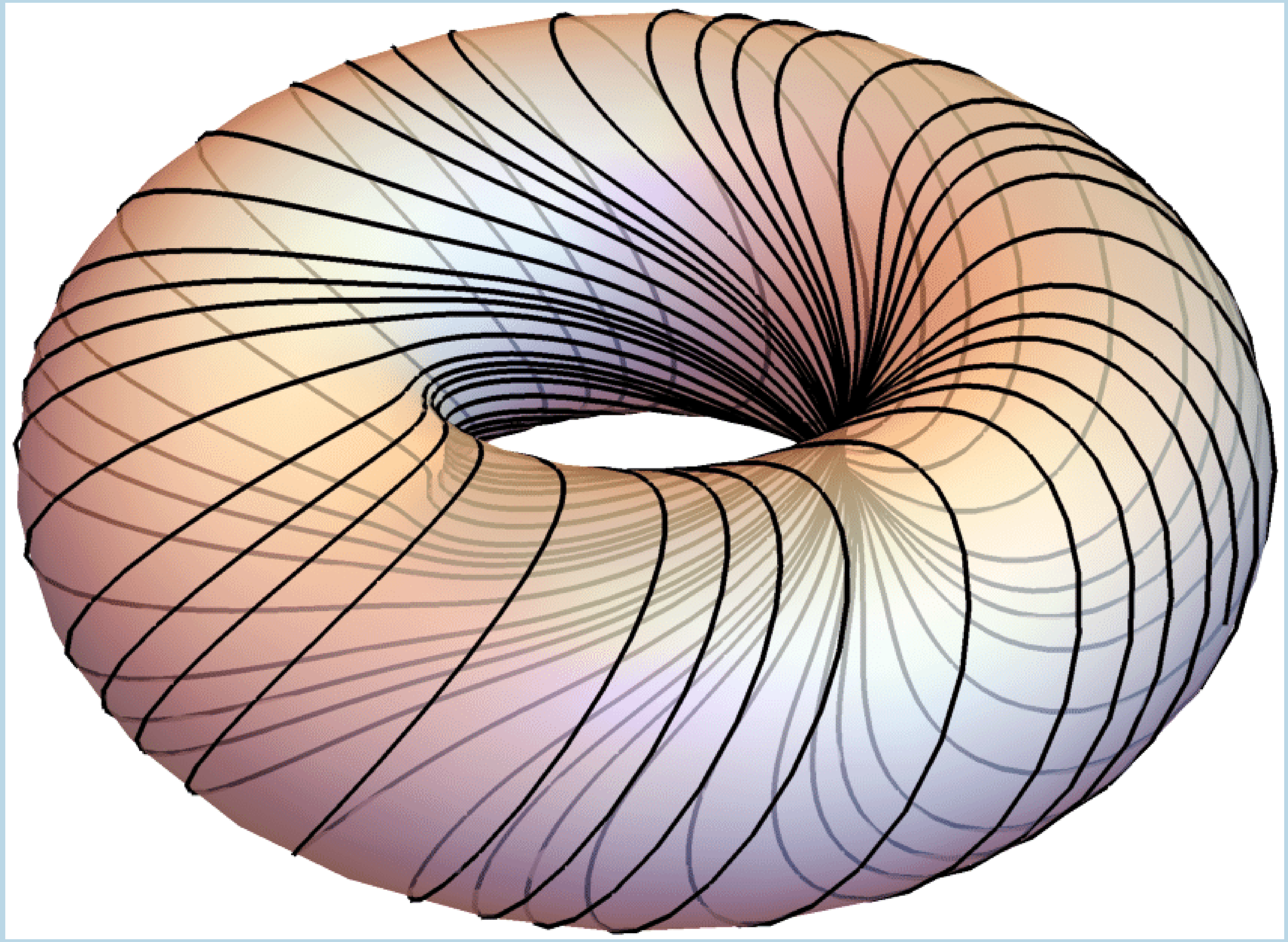
Definition 4. *The **structure functions** of β are the functions c_{ij}^k so that*

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k$$

A Theorem that Ensures Positive Scalar Curvature

Definition 5. *Let Y be a sub-bundle of constant rank of a smooth manifold M. Y is called **involutive** if for all vector fields $U, V \in \Gamma(M, Y)$, we also have $[U, V] \in \Gamma(M, Y)$.*

Theorem 3 (Frobenius). *If Y is a subbundle of constant rank on a smooth manifold M, then Y is involutive if and only if Y is the tangent space of a foliation of M.*



Definition 6. *Let (M, g) be a Riemannian manifold. We say that g is **nearly bundle-like** with respect to a vector sub-bundle Y of M if there exists a local orthonormal frame in an open neighborhood around every point $x \in M$ so that the inequality $\frac{1}{2}S_2 > 2e_\gamma(c_{\gamma j}^j) - c_{\alpha k}^k c_{\alpha j}^j - 2c_{\alpha k}^k c_{\alpha\beta}^\beta - \frac{1}{2}c_{i\beta}^k c_{k\beta}^i - c_{\gamma\beta}^i c_{i\beta}^\gamma - \frac{1}{2}c_{\gamma j}^i c_{\gamma j}^i$, where S_2 is the scalar curvature of M restricted to Y, holds true.*

Corollary 1. *If, in addition to the hypothesis above, the sub-bundle Y is also involutive, then the formula above becomes*

$$\begin{aligned} S = {} & \frac{1}{2}S_1 + \frac{1}{2}a^2 S_2 \\ & + \left(2e_\gamma(c_{\gamma j}^j) - c_{\alpha k}^k c_{\alpha j}^j - 2c_{\alpha k}^k c_{\alpha\beta}^\beta - \frac{1}{2}c_{i\beta}^k c_{k\beta}^i - c_{\gamma\beta}^i c_{i\beta}^\gamma - \frac{1}{2}c_{\gamma j}^i c_{\gamma j}^i\right) a^2 \\ & + \left(2e_k(c_{k\beta}^\beta) - c_{i\gamma}^\gamma c_{i\beta}^\beta - 2c_{i\gamma}^\gamma c_{ij}^j - \frac{1}{2}c_{\gamma j}^\alpha c_{\alpha j}^\gamma - c_{\gamma j}^i c_{ij}^\gamma - \frac{1}{2}c_{k\beta}^\alpha c_{k\beta}^\alpha\right) \\ & - \frac{1}{4}\left(\frac{1}{a^2}\right)(c_{kj}^\alpha c_{kj}^\alpha). \end{aligned}$$

If we also have that g is nearly bundle-like with respect to Y, then there exists a constant $M > 0$ so that for all $a \geq M$, $S_2 > 0$ implies that $S > 0$.