Two-Dimensional Discrete Predator-Prey System

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Abstract

- We show that the following discrete two-dimensional logistic predator-prey dynamical system with two parameters undergoes a Neimark-Sacker bifurcation under certain conditions. Our evidence includes numerical computations of orbits and bifurcation diagrams.

- $x_{n+1} = f(x_n, y_n) = k_1(1-x_n)x_n + x_n y_n$
- $y_{n+1} = g(x_n, y_n) = k_2(1-y_n)y_n - x_n y_n$

Process

- The first step in studying a discrete dynamical system is to calculate its fixed points. A fixed point (x^*, y^*) exists if $f(x^*, y^*) = x^*$ and $g(x^*, y^*) = y^*$. The fixed points for the predator-prey system are (0,0), $((k_1-1)/k_1, 0)$, $(0, (k_2-1)/k_2)$, and $((k_1k_2-1)/(k_1k_2+1), 1 - (2k_1)/(k_1k_2+1))$. The fourth of these points is the only one that must be studied in two-dimensions if it has nonzero coordinates. We focus on the fourth point when its coordinates are positive, since negative populations are meaningless.
- The second step is to analyze the stability of the fourth equilibrium point. This is achieved by calculating the eigenvalues of the Jacobian matrix at the fourth point. The Jacobian matrix i

$$J_f = \begin{bmatrix} k_1 - 2k_1x^* + y^* & x^* \\ -y^* & k_2 - 2k_2y^* - x^* \end{bmatrix}$$

- After finding the above matrix, we find the determinant of the equation J_f -Aλ and solve for the eigenvalues $\lambda_{1,2}$. These eigenvalues will determine the stability or instability of a fixed point.
- If the eigenvalues are real, the fixed point is stable when $\max\{|\lambda_1|, |\lambda_2|\} < 1$, and unstable if one or both eigenvalues have magnitude(s) greater than 1. If the eigenvalues are complex, the fixed point is stable when $|\lambda_{1,2}| < 1$ and unstable when the magnitude is greater than 1.
- Neimark-Sacker bifurcations occur when the complex eigenvalues have a magnitude equal to 1. After the bifurcation, an invariant curve forms around the fixed point, which can either attract or repel nearby orbits (orbits being the population values over time). A supercritical Neimark-Sacker bifurcation produces a stable invariant curve, while a subcritical Neimark-Sacker bifurcation results in an unstable invariant curve.

Bifurcation Diagrams

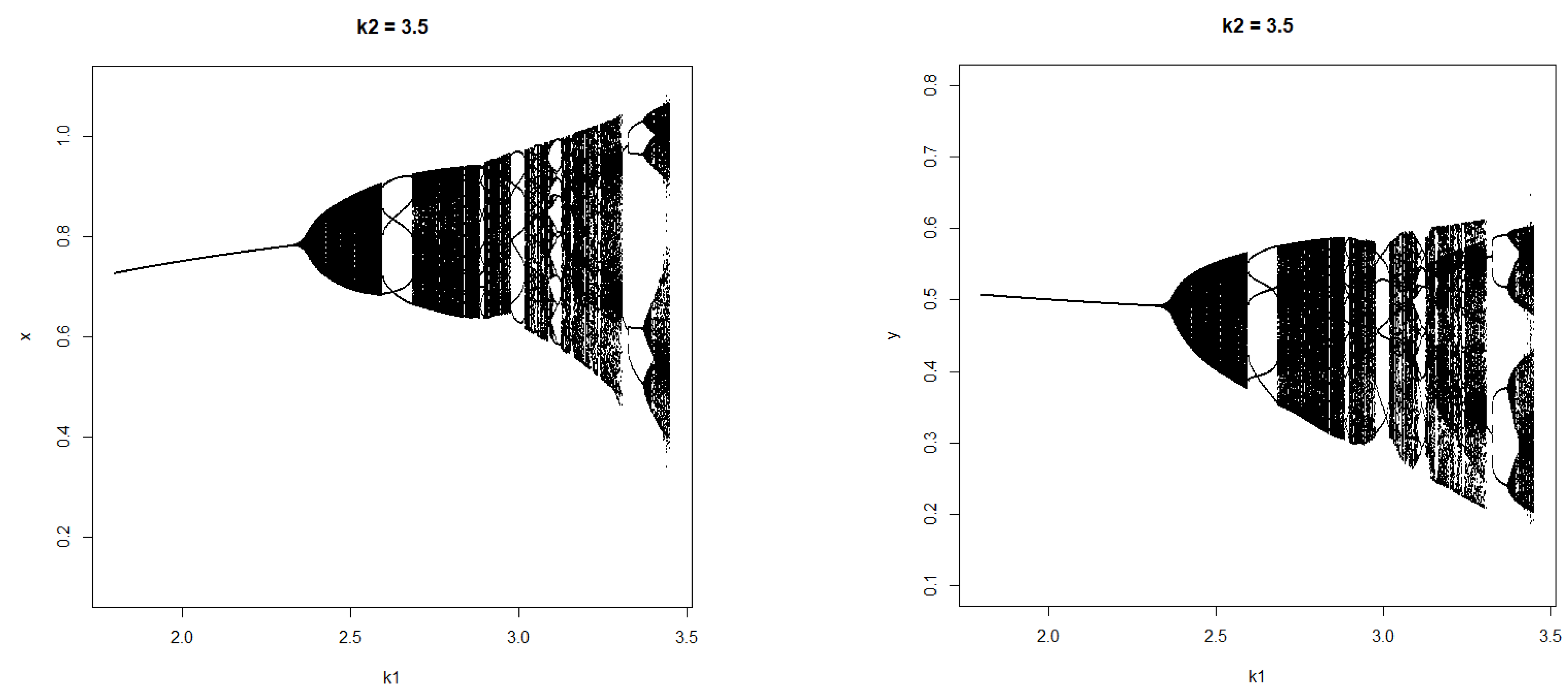

Figure 1: Bifurcation diagrams plotting the long-term behavior for the predator (left) and prey (right) populations as k_1 is varied over an interval, while k_2 is fixed at 3.5. The Neimark-Sacker bifurcation occurs when k_1 is approximately 2.3705. Beyond this value, a funnel shape is produced in both graphs; this is the invariant curve that has formed around the fourth fixed point. Various *n*-cycles, such as a 5-cycle, a 7-cycle, a 2-cycle, and a 4-cycle, also show up throughout the funnel.

Videos

Videos were made to better visualize how changes in the growth parameter k_1 leads to changes in the long-term behavior of the two populations. In the first video, the populations are plotted in two dimensions, where each frame has a different k_1 value. The second video shows the same behavior as the first, but plots k_1 along the *z*-axis, producing a 3D bifurcation diagram.

k_1-lapse video

3D bifurcation diagram

Conclusions

The discrete predator-prey dynamical system exhibits a great variety of orbit behaviors, including the Neimark-Sacker bifurcation and its resulting invariant curve, chaotic regions, strange attractors, and periodic orbits of various periods.

Future Work

The various invariant curves, as seen in the k_1-lapse video to the left, may require further research, as they can occur under both complex and real eigenvalues, as opposed to the Neimark-Sacker bifurcation. The fractal dimension of the strange attractors that form around the fourth fixed point may also be calculated in the future.

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