Geodesic Nets construction using Genetic Algorithm

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Introduction

Geodesics are significant objects and a major topic in differential geometry. They are "straight" curves on surfaces that can locally represent the shortest path between two points.

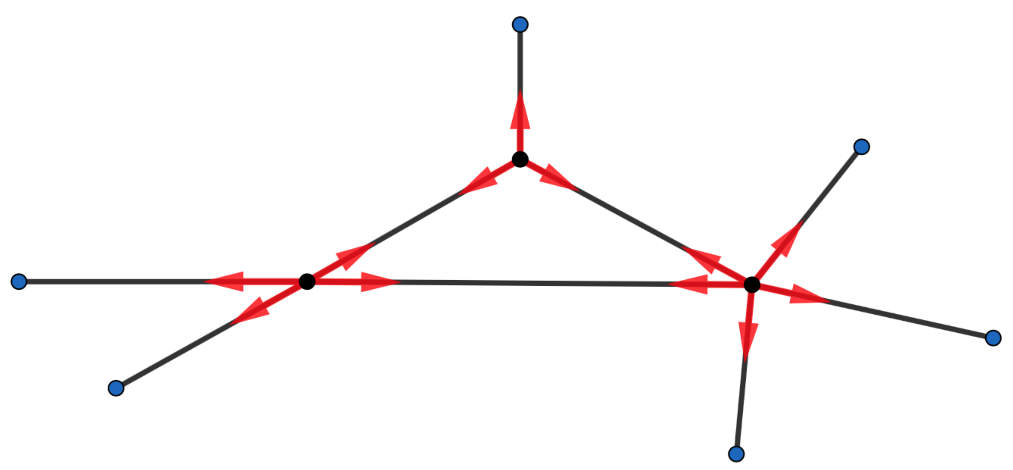
In this research, we employ the genetic algorithm, an optimization method in classical Artificial Intelligence, to construct a geodesic net on closed surfaces. A geodesic net is a network that connects multiple points with the shortest curves while ensuring that each point is "*balanced*" and stretched equally by its neighbors through those curves.

Geodesic Net Definition

Definition 1. *Let S be a (possibly empty) finite set of points in a Riemannian manifold M. A geodesic net on M consists of a finite set V of points of M (called* ***vertices****) that includes S and a finite set E of non-constant distinct geodesics between vertices (called* ***edges****) so that for every vertex $v \in V - S$, the following balancing condition holds:*

- *Construct a unit tangent vector from v towards the endpoint of each edge connected to v. Then, the sum of all these tangent vectors must be equal to $\vec{0}$.*
- *Edges may not intersect or self-intersect.*

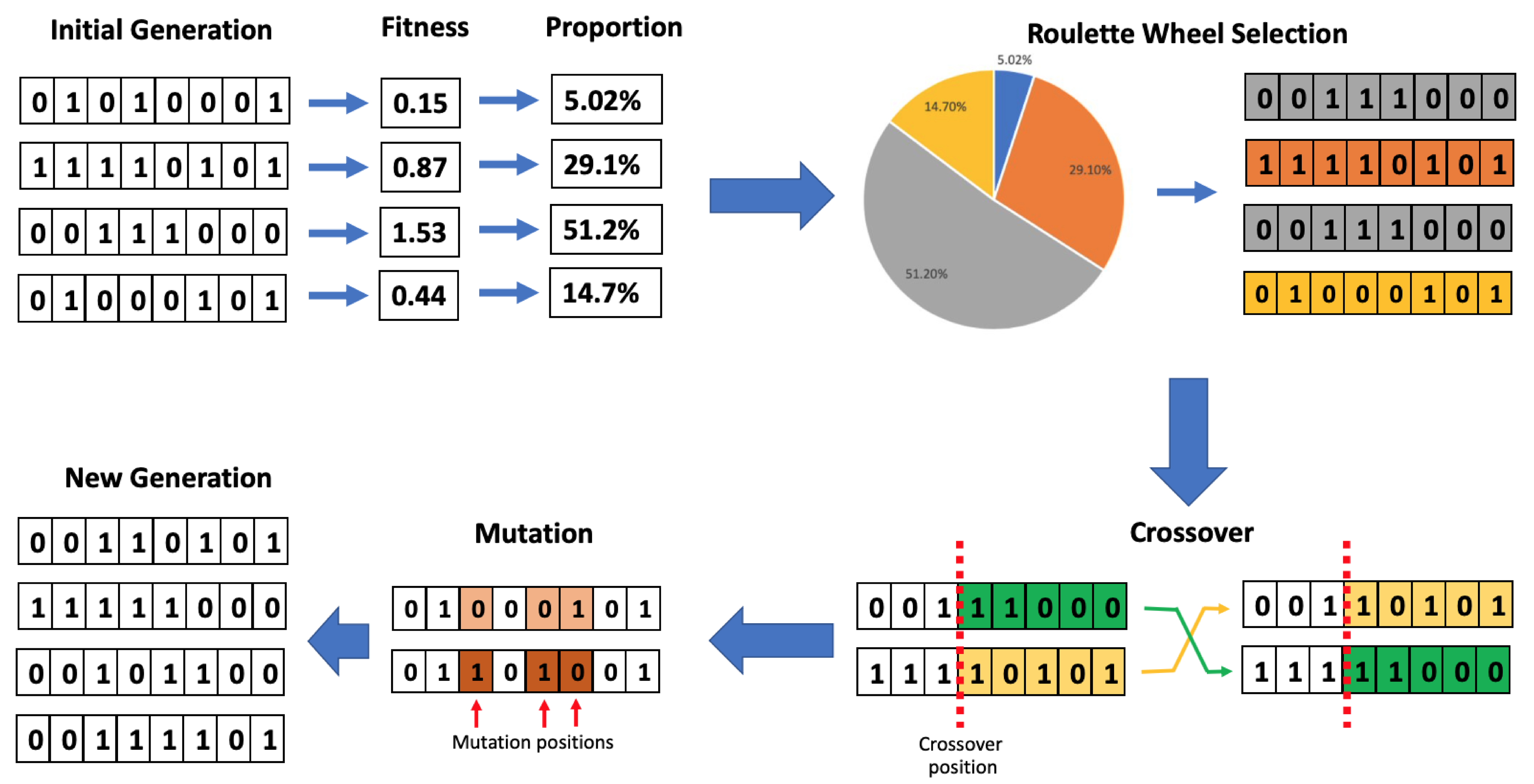
The vertices in S are called ***unbalanced****, and the vertices in $V - S$ are called* ***balanced****. For each unbalanced vertex v, the sum of all unit tangent vectors to adjacent vertices of v is not necessarily equal to $\vec{0}$.*



The figure above shows a geodesic net in Euclidean space. The black vertices are balanced and the blue vertices are unbalanced.

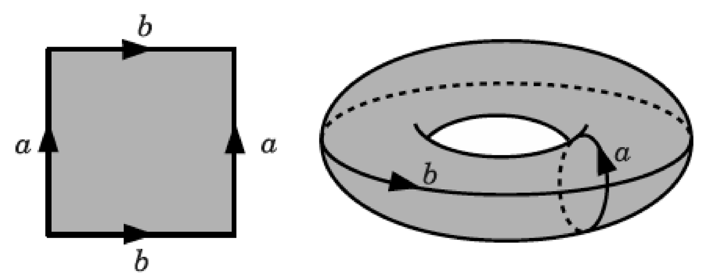
Genetic Algorithm

Genetic Algorithm (GA) is a type of evolutionary algorithm (EA) that mimics the process of natural selection. It is used to solve optimization and search problems by employing biologically-inspired techniques such as mutation, crossover, and selection to generate optimal solutions.

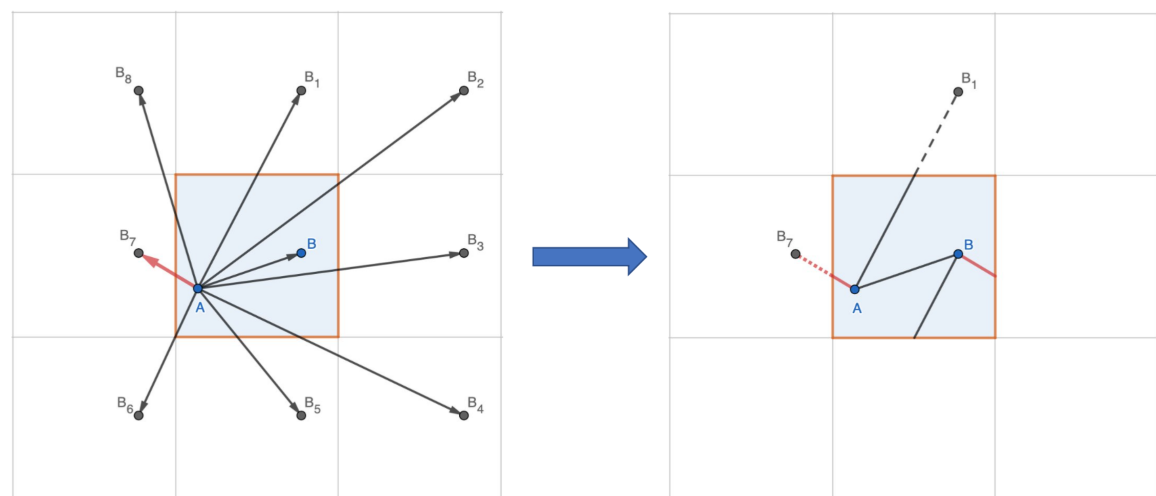


Problem Set-up

In this research, we construct a geodesic net of five balanced vertices A_1, A_2, A_3, A_4, A_5 on **a square unit flat torus**. A **position** contains five $x, y-$*coordinates* for five points such that $A_i = (x_i, y_i)$ for $x_i, y_i \in [0, 1]$.



Given two vertices A and B on the flat torus, in the picture, there are nine straight paths that connect A to B. We only construct the shortest segment among them. Then, we take the unit vector from A to B on that segment.



Construction Algorithm

Initially, we generate randomly 1000 chromosomes, which are the sequences of bits of length 100. Each chromosome represents a position of A_i's by using 10 bits for each coordinate x_i or y_i.

$$\underbrace{|1|0|\cdots|1|1|}_{b_1}\underbrace{|0|1|\cdots|0|1|}_{c_1}\cdots\underbrace{|1|1|\cdots|1|1}_{b_5}\underbrace{|1|0|\cdots|1|1|}_{c_5}$$

Next, we calculate the value for x_i's and y_i's by

$$x_i = \frac{D(b_i)}{2^{10}-1} \text{ and } y_i = \frac{D(c_i)}{2^{10}-1},$$

where $D(e)$ is the integer associated to the 10 bits e. Then, we calculate the **fitness** of each chromosome:

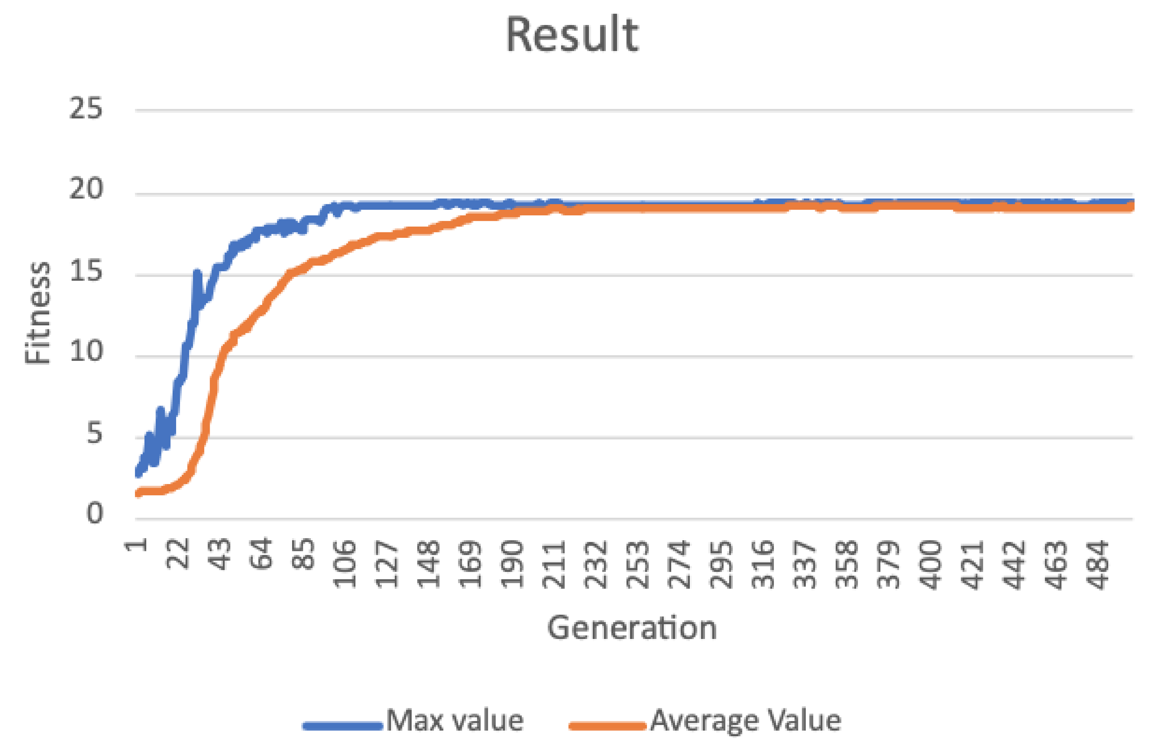
1. For each point A_i, calculate the sum of all unit tangent vectors as $\vec{v_i}$.
2. Then, the fitness function is

$$f = \frac{20}{\sum_{i=1}^{5} ||\vec{v_i}|| + 1}$$

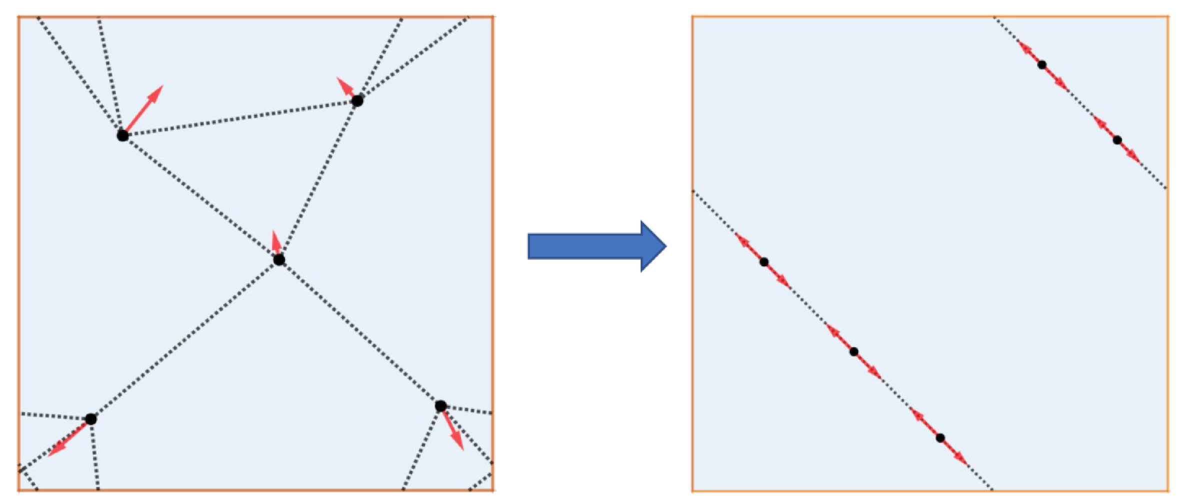
We apply the Genetic Algorithm for 500 generations and get the position with maximum fitness.

Results

The maximum fitness in the first generation, which includes 1000 random positions, is quite low. After running the algorithm for 500 generations, the maximum and the average fitness of chromosomes in the last generation is almost 20, which mostly approaches the maximum of the fitness function.



Also, we can see the average and maximum values converge through generations, which mean the good chromosomes are selected and dominant the population.



First generation 500th generation

Future Work

In our future research, we plan to investigate geodesic nets with a larger number of points on various surfaces with different Riemannian metrics. Additionally, we aim to implement other computational algorithms such as Monte Carlo simulation, gradient descent, or deep learning models to enhance the performance efficiency of our investigations. These approaches can provide valuable insights into the behavior and properties of geodesic nets, and potentially accelerate the computations.