Cheeger Constant: A Measure of Connectivity of Graphs

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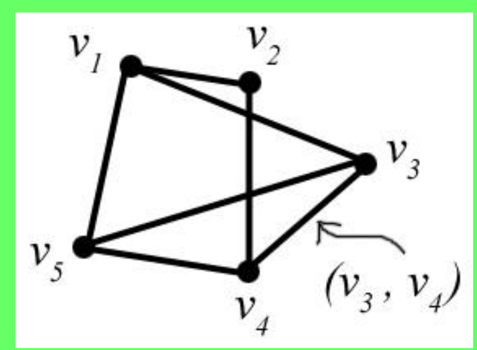
Abstract: Cheeger constant shows the connectivity of a graph and gives the best possible way to cut a graph. A smaller Cheeger constant indicates that a larger area would have to be removed when trying to disconnect a graph, and hence the more stable a network would tend to be if a component failed. I looked at possible values for the Cheeger constant and found that the constant would have to be between zero and one. I computed the value of the Cheeger constant for cycle graphs, complete graphs, bi-partite graphs, and cube graphs and looked at how the Cheeger constant changes when one removes an edge from a graph or collapses an edge to a vertex.

The Cheeger constant relates to the second-smallest eigenvalue of a symmetric matrix A, where A equals the Laplacian of a graph G, because when one arrives at a value for the Cheeger constant of a graph then one comes to an estimate of that graph's second-smallest eigenvalue. The second smallest Laplacian eigenvalue has been called the most important information contained in the spectrum of a graph, and it gives the algebraic connectivity of a graph. The Laplacian matrix appears in the theory of electric current and flows, while the Laplacian eigenvalues determine the kinematic behavior of a liquid flowing through a system of communicating pipe.

Definitions:

Deg V: the number of edges going from a vertex to any other vertex. Ex: The degree of v_1 is 3.



Vol S: the sum of the degree of all the vertices in S.

S⊆V: Let S be a subset of V, then each of the vertices in S must be in V.

E(S,S^c): the number of edges that go from a vertex in S to a vertex in S^c.

S^c: the complement of set S, contains every vertex not contained in S.

Isomorphic sets: the number of edges joining each pair of vertices in a set equals the number of edges joining the corresponding pair of vertices in another set.

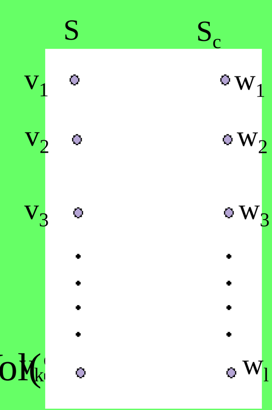
Cheeger Constant of a Graph: shows the connectivity of a graph and gives the best possible way to cut a graph.

The Cheeger constant equals:
h_G= min{abs(E(S,S^c))/{min(Vol(S), Vol(S^c))}}= h_G(S) for every possible value of S unless they are isomorphic.

Computation: Let S={v_1, v_4, v_5} and {v_2, v_3} from the graph at the top.
Then E(S,S^c) = 2 + 2 + 1 = 5, Vol(S) = 3 + 3 + 3 = 9, Vol(S^c) = 3 + 2 = 5
Hence, h_G(S) = 5/min(9, 5) = 1. Then h_G would be the minimum of every possible value of h_G(S).

We found that the Cheeger constant would have to be between zero and one.

Outline of Proof:
We know by definition of h_G(s) that $h_G(s) \geq 0$.
G – connected graph
S = {v_1,v_2,…,v_k}, S^c = {w_1,w_2,…,w_l}
Vol(S) = deg(v_1) + deg(v_2) + … + deg(v_k)
Vol(S^c) = deg(w_1) + deg(w_2) + … + deg(w_l)

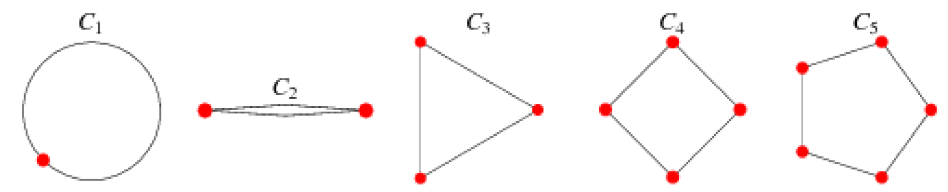


By definition of E(S,S^c), the value for the edges connecting S to S^c has a maximum value of the minimum of the Vol.

Since E(S,S^c) ≤ min(Vol(S), Vol(S^c)), so $h_G(s) \leq 1$.

When one collapses a vertex to another vertex, the Cheeger constant may increase on decrease depending on the case. Now when one removes an edge between two vertices v_1 and v_2, if h_G occurs when either v_1 or v_2 is in S and the other is in S^c, then the Cheeger constant will decrease with the removal of the edge. Otherwise the Cheeger constant may increase or decrease.

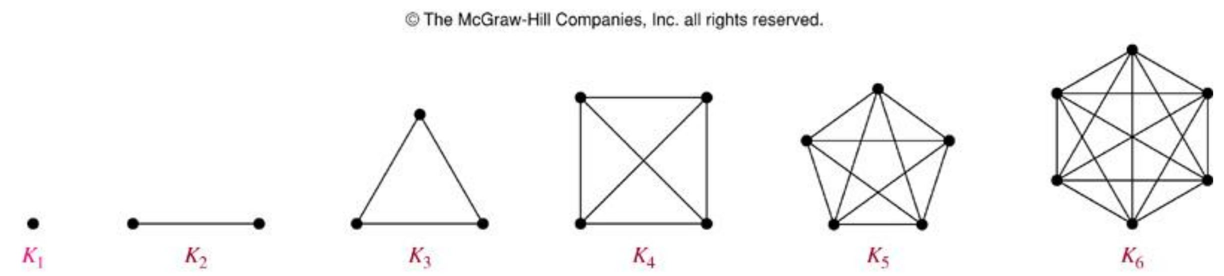
Cycle Graphs: a graph consisting of a single cycle of vertices and edges. Notation for cycle graphs: C_n.



Results: Let n be the number of vertices in a cycle graph. We showed in general that for cycle graphs with an even number of vertices, h_{c_n} will have a minimum value of (2/n). For a cycle graph with an odd number of vertices, h_{c_n} will have a minimum value of (2/(n – 1)).

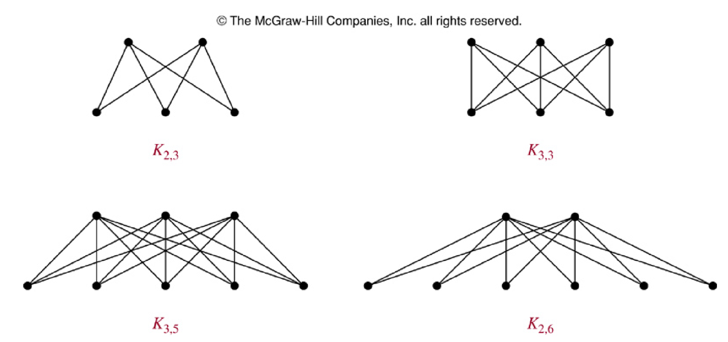
Complete Graphs: a graph in which every vertex is joined to each of the others by exactly one edge.

Notation for complete graphs: K_n.



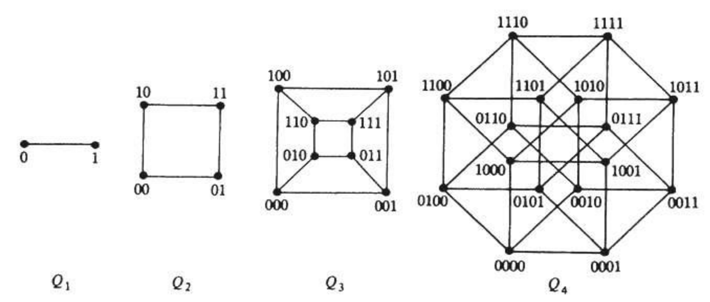
Results: Let n be the number of vertices in a complete graph. We showed in general that for complete graphs with an even number of vertices, h_{k_n} will have a minimum value of (n/(2n – 2)). For a complete graph with an odd number of vertices, h_{k_n} will have a minimum value of ((n+1)/(2n – 2)). The smallest value for the Cheeger constant for both the cycle and the complete graphs occurs when one divides the vertices up evenly as possible between the subset S and S_c.

Bipartite Graphs: a graph whose set of vertices can be split into two subsets A and B in such a way the each edge of the graph joins a vertex in A and a vertex in B. Notation for bipartite graphs: $K_{m,n}$.



Results: For the bipartite graphs $K_{m,n}$ where m and n are even, then the Cheeger constant equals ½. For m and n odd, we have a notion of where the Cheeger constant occurs when n is a multiple of m, otherwise where the Cheeger constant occurs is more difficult to pin-down.

N-Cube Graphs: a graph with 2^n vertices, where each vertex receives a binary label with n digits till every possible binary label has been assigned to a vertex. No two vertices share the same binary labeling, and vertices are only adjacent to each to each other if there binary labels differ by only one digit. Notation for n-cube graphs: Q_n.



Results: For the n-cube graphs Q_n, the Cheeger constant equals 1/n.